DESIGN CONTOURS FOR LOAD COMBINATIONS: GENERALIZING INVERSE FORM METHODS TO DYNAMIC PROBLEMS

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Abstract: In static structural reliability problems involving a set $X$ of random variables, it is common to relate the failure probability $p_f$ to the event that a safety margin $M = g(X)$ becomes negative. For dynamic responses to a combination of stationary load processes $X(t)$, we commonly seek $v$, the mean out-crossing rate by $X(t)$ into the unsafe region $g[X(t)] < 0$. Standard FORM (First-Order Reliability) methods can be applied to both the static and dynamic problems (e.g., [1]).

A challenge comes in the design context, when a target $p_f$ or $v$ is specified and a particular design variable (e.g., capacity) is sought. In the static case, inverse FORM methods have been suggested (e.g., [2,3]). The resulting "design contour" may be viewed, in the transformed space of standard Gaussian variables $U$, as the locus of all possible FORM design points that can give the desired $p_f$ value. These contours have been used for the static problem [4,5,6,7] to determine "design contours" of wind-wave site characteristics (e.g., $N$-year conditions) for design, wave-tank experiments, etc.

We seek here to develop analogous design contours for the dynamic problem. One proposal in this case uses elliptical contours in $U$-space [8]. We develop alternative contours based on FORM principles, again defining the contour as a locus of all possible FORM design points. We apply these contours to study a load combination problem involving the midspan vertical bending moment of a ship. Finally, we present some data on the accuracy of several methods of analysis, and the effect of modeling choices for $X(t)$.

1 Introduction

In structural reliability, it is common to relate the failure probability $p_f$ to a limit state (or failure surface) $g(X)$ involving a vector $X$ of random variables. The probability distribution of $X$ is assumed to be known and failure is presumed to occur if $g(X) < 0$. In many practical situations the random quantities of interest relate to the demand $d$ on the structure, while the capacity $c$ is determined by the designer. In this situation $c$ is treated as deterministic and the limit state can be written as $g(X) = c - d(X)$. This study will be written using this simplified formulation. In cases of dynamic responses to a load process $X(t)$, we commonly use a Poisson model to estimate the probability of failure, $p_f(T)$, during a specific time interval $T$. This only requires knowledge of the mean out-crossing rate by $X(t)$ into the unsafe region $d[X(t)] > c$, denoted as $v_d[X(t)](c)$.

Standard FORM (First-Order Reliability) methods can be used to approximate $p_f$ for given probability distributions of $X$ for either the static or dynamic situation (e.g., [1]). First the Rosenblatt transformation is used to map $X$ into a vector $U$ of independent standard Gaussian random components. This maps the failure condition of $d(X) = c$ into a corresponding limit state $\hat{d}(U) = c$. In either the static or the dynamic problem, interest is focused on the most likely point on this limit state. In the standard Gaussian $U$-space this point $u^*$ is simply the point on the limit state that is closest the origin. The FORM analysis then approximates the limit state by its tangent plane at the point $u^*$, thus simplifying the calculation of the probability of failure. In the static case the FORM approximation of $p_f$ depends only on the distance of $u^*$ from the origin, denoted as $\beta(u^*)$, and in the dynamic case it depends on both $\beta(u^2)$ and crossing rate information.

The design problem is the inverse of the above procedure. That is, there is now a given value of $p_f$ but at least some aspects of the capacity $c$ are
unknown, since they depend on design parameters that have not yet been chosen. In the static case, inverse FORM methods have been suggested (e.g., [2,3]). The resulting "design contours" may be viewed as the locus, in \( \mathbf{U} \) space, of all possible design points that produce the desired \( p_f \). These contours are simply spheres in \( \mathbf{U} \) space for the static problem, since \( p_f \) for any point \( \mathbf{u} \) depends only on the distance from the origin, \( \beta(\mathbf{u}) \). These contours have been used \([4,5,6,7]\) to derive "design contours" of wind-wave site characteristics (e.g., \( N \)-year conditions) for design, wave tank experiments, etc.

We seek here to extend the inverse FORM method to develop analogous contours for dynamic problems. One prior proposal for this situation uses elliptical contours in \( \mathbf{U} \) space \([8]\). We develop alternative contours by using FORM principles to identify the locus of all possible design points having the crossing rate that gives the specified probability of failure. Since the FORM estimate of the crossing rate depends not only on \( \beta(\mathbf{u}) \) but also on given information about the crossing rates of \( \mathbf{X} \), the \( \mathbf{U} \) space contours are generally not circular. Finally, we apply these contours to study a load combination problem involving the midspan vertical bending moment of a ship.

## 2 Mathematical Formulation

The notation used here is based on using capital letters for random variables or processes and lower case letters for deterministic quantities, or "dummy values" of these random terms. The common convention of using bold face for vector quantities is also employed.

The equations for the probability of failure, as described above, are

**Static case:**

\[
p_f = P[d(\mathbf{X}) > c] \quad (1)
\]

**Dynamic case:**

\[
p_f = 1 - \exp\left(-T \cdot \nu_d(X)\right)(c) \quad (2)
\]

in which the dynamic result is based on the Poisson approximation for the probability of a crossing by the stationary time history \( X(t) \) into the failure region during a time interval of duration \( T \).

We will write the Rosenblatt transformation and its inverse as

\[
\mathbf{U} = f(\mathbf{X}) \quad \mathbf{X} = h(\mathbf{U}) \quad (3)
\]

This mapping of \( \mathbf{X} \) into the standard Gaussian \( \mathbf{U} \), also maps the limit state \( d(\mathbf{X}) = c \) into \( \hat{d}(\mathbf{u}) = d[h(\mathbf{u})] = c \). The design point (or "characteristic point") \( \mathbf{u}^* \) is the most likely point on the limit state, which means it gives the minimum value of \( \beta(\mathbf{u}) = |\mathbf{u}| \): 

\[
\beta(\mathbf{u}) \geq \beta(\mathbf{u}^*) \quad \text{for all } \mathbf{u} \text{ with } \hat{d}(\mathbf{u}) = c \quad (4)
\]

The FORM approximation then replaces the limit state in \( \mathbf{U} \)-space with the plane that is tangent to the limit-state surface at \( \mathbf{u}^* \). With this approximation, the failure condition of \( \hat{d}(\mathbf{u}) > c \) can be written as \( Z > \beta(\mathbf{u}^*) \) in which \( Z \) is a unit-variance scalar Gaussian random variable that is a linear function of \( \mathbf{U} \). One way to think of \( Z \) is as the projection of \( \mathbf{U} \) onto the line from the origin to \( \mathbf{u}^* \), which can be written using direction cosines or using a vector inner product:

\[
Z(\mathbf{U}) = \sum U_i \cos(\theta_i) = \mathbf{U}^T \mathbf{u}^* / \beta(\mathbf{u}^*) \quad (5)
\]

in which \( \theta_i \) is the angle between the vector \( \mathbf{u}^* \) and the \( U_i \) axis. For this standard Gaussian measure of demand, (1) gives

**Static case:**

\[
p_f = \Phi[-\beta(\mathbf{u}^*)] \quad (6)
\]

in which \( \Phi \) denotes the cumulative distribution of the standard Gaussian random variable. Similarly the dynamic probability of failure is based on the rate of crossing of the FORM plane by \( \mathbf{U}(t) \). This is the same as crossings by the unit-variance Gaussian time history \( Z(t) \equiv Z(\mathbf{U}(t)) \). Thus

**Dynamic case:**

\[
p_f = 1 - \exp\left(-\nu_Z(\mathbf{u}^*)T \right) \quad (7)
\]

Finding the crossing rate of \( Z(t) \) in the Gaussian \( \mathbf{U} \) space requires knowledge of the joint distribution of \( Z(t) \) and \( \dot{Z}(t) \). Some caution is needed at this point. The obvious choice is to say that \( Z(t) \) is a Gaussian process, for which \( Z(t) \) and \( \dot{Z}(t) \) are jointly Gaussian. We will use this assumption in the following FORM analysis, but it is not directly implied by the Rosenblatt transformation in (3). All that is assured by the Rosenblatt transformation is that \( \mathbf{U}(t) \) is a standard Gaussian random variable for any value of \( t \). It does not assure that \( \mathbf{U}(t_1) \) and \( \mathbf{U}(t_2) \) are jointly Gaussian for \( t_1 \neq t_2 \), which is required in order to have \( \mathbf{U}(t) \) be Gaussian, and jointly Gaussian with \( \mathbf{U}(t) \)--that is to have \( \mathbf{U}(t) \) be a standard Gaussian process. The special case in which \( \mathbf{U}(t) \) is a standard Gaussian process requires that \( \mathbf{X}(t) \) is what is called a translation process \([9]\), which only means that (3) relates it to a standard Gaussian process \( \mathbf{U}(t) \).

For a standard Gaussian process we know that

\[
\nu(u) = \nu_0 e^{-u^2/2} \quad (8)
\]

in which \( \nu_0 \) is the mean crossing rate of zero. Thus, using the translation process assumption in (7) gives
Dynamic case:  
\[ p_f = 1 - \exp \left( -v_{Z(t)}(0) T e^{-\beta^2(u^*)/2} \right) \]  

The design problem is the inverse of the above procedure. That is, there is now a given value of \( p_f \) and the problem is to find the design point \( x^* \) out of all the \( x \) values that give this value of \( p_f \). For this problem it is useful to introduce the idea of a contour in \( X \)-space or \( U \)-space that contains all points with the specified value of \( p_f \). Of course \( u^* \) is one point on that contour in \( U \)-space, and the problem is to find this point \( u^* \) that not only is on the contour but also is the most likely point on a limit-state surface passing through that point.

Of course the value of the capacity \( c \) is not known \textit{a priori} in the design problem, so neither is the limit state. Thus, one can picture a multi-dimensional limit-state map in either \( X \)-space or \( U \)-space with each limit-state surface corresponding to a different value of \( c \). One such contour surface will pass through each \( u \) point. Since \( u^* \) is the most likely point on the appropriate limit state, it is necessary that the limit-state surface is locally orthogonal to the vector from the origin to \( u^* \). This orthogonality condition identifies \( u^* \), and this point is then mapped back into the \( X \)-space to give the so-called "design point" as \( x^* = h(u^*) \) and the design value of the capacity \( c \).

**Static Inverse FORM**

The demand quantity in the inverse problem may be of various forms, such as a load, a load effect, or a structural response. In offshore applications it is commonly a global load such as base shear or overturning moment on an offshore platform [2]. For a floating platform it may be a displacement, such as surge motion [4], and for the ship example that follows, it is the midspan vertical bending moment. The object is to find an appropriate design value of this quantity based on a given probability of failure \( p_f \).

For simplicity we will consider a case in which \( X \) and \( U \) have only two dimensions. Then any point \( u \) can be written in a polar notation form of

\[ u = (u_1, u_2) = \beta(u)[\cos(\theta), \sin(\theta)] \]  

in which \( \theta = \tan^{-1}(u_2/u_1) \). From (6) we then have the constant- \( p_f \) contour as the set of \( u \) values satisfying

\[ \beta(u) = \Phi^{-1}(p_f) \]  

and (10) and (11) give a parametric representation of the contour by considering \( \theta = [0, 2\pi] \).

![Figure 1: An example in U-space when \( \beta = 3 \) (\( p_f = 1.3 \times 10^{-3} \)).](image)

Any limit state satisfying this condition must be tangent to the circle with radius \( \beta = 3 \).

Figure 1 illustrates the situation. It shows both a true (nonlinear) limit state \( \hat{g}(u) = c - \hat{d}(u_1, u_2) \) for a particular value of \( c \) and a linearized approximation, tangent to the true limit state at the point of maximum probability (nearest the origin). The expression in (11) is based on the linearized approximation of the limit state, so is not exactly correct for the true limit state. The plot is for \( \beta = 3 \), and hence \( p_f = 1.3 \times 10^{-3} \). Now we will consider the design problem with \( p_f = 1.3 \times 10^{-3} \). Since the capacity of \( c \) is unknown at this point in a design problem, one can picture other \( g(u_1, u_2) = 0 \) limit-state curves being added to Fig. 1, with one such curve passing through every point on the \( \beta = 3 \) contour. The condition of local orthogonality between \( u^* \) and the limit-state, makes it clear that \( u^* \) must be a point of tangency between the contour and the appropriate limit-state. This point must give the largest value of the demand on the contour. Thus, to find the controlling design situation for this value of \( p_f \) we only need to search the \( \beta = 3 \) contour for the point with the largest value of \( \hat{d}(u_1, u_2) \). If the system is designed with a capacity equal to this demand, then the design will be deemed adequate for all other demands with the same value of \( p_f \).
10,000 years). These are then transformed into the 

$$H_s - T_p$$

contours shown in the figure. The structural analyst then need only search for the maximum load/response demand over the contour with the desired return period. That is, finding the maximum of 

$$d(u_1, u_2)$$

is the same as finding the maximum of 

$$d(x_1, x_2).$$

If the demand depends on more than two \( x_i \) components, the contour logic extends directly; i.e., \( \beta \) is now the radius of an \( N \)-dimensional sphere. Clearly, though, the convenient visual nature of environmental contours is best served when \( N = 2 \).

**Dynamic Inverse FORM**

The above theory is a simplified approximation of a real problem, inasmuch as each \( X_i \) is assumed to be constant throughout a period of analysis; e.g., there is one value throughout every \( T = 3 \)-hour seastate. Limited dynamic effects can be included by considering only the maximum value of a single time-varying response (while other \( X_i \) reflect slowly varying environmental parameters, such as \( H_s \) and \( T_p \)).

The general load combination problem, however, will include multiple time-varying components, \( X_i(t) \), each typically with different frequency content. This is the problem we seek to address here.

It has become customary to define the “characteristic” (or most probable) response as the level with an exceedance probability of \( p_f = 1 - e^{-1} \). This is the Poisson estimate of the probability of failure if one failure is expected during the time interval \( T \) in (2). From (2) and (8) we find the characteristic value

$$\beta_{cr} = [2 \ln(N)]^{1/2}$$

in which \( N = \nu_0 T \) is the average number of response cycles over the duration \( T \) of interest. This is a commonly cited result for standard normal responses.

We now consider a demand process \( Z(t) \) that is a linear combination of two independent processes, each transformed and standardized as in (3) in terms of standard normal processes \( U_1(t) \) and \( U_2(t) \). One parametric form for the linear combination is

$$Z(t) = U_1(t) \cos(\theta) + U_2(t) \sin(\theta)$$

(13)

Because \( U_1 \) and \( U_2 \) are both standard Gaussian, any \( (u_1, u_2) \) contour that could contain the characteristic extreme must approach the limiting values of \( \beta_1 = [2 \ln(N_1)]^{1/2} \) and \( \beta_2 = [2 \ln(N_2)]^{1/2} \) when \( u_2 = 0 \) and \( u_1 = 0 \), respectively. Here \( N_1 \) and \( N_2 \) are the average number of cycles of each response over the duration \( T \) of interest.

What remains in question, of course, is how the contour should be defined in intermediate cases. Leira [8] has suggested an ellipsoidal contour:

$$\left( \frac{u_1}{\beta_1} \right)^2 + \left( \frac{u_2}{\beta_2} \right)^2 = 1; \quad \beta_i = [2 \ln(N_i)]^{1/2} \text{ for } i = 1, 2 \quad (14)$$

and the following sections will provide an alternate approach.

We return now to the formalism of FORM. Recall that this method seeks results that are exact within the limitations of linearized limit state in U-space and the Poisson approximation of the crossing rate. Thus, the starting point is again the demand quantity \( Z(t) \) in (13). We consider \( U_1(t) \) and \( U_2(t) \) to be standard Gaussian processes, so \( Z(t) \) is also a standard Gaussian process. This implies that we can again use the standard result from (12) for its characteristic value:

$$z_{cr} = [2 \ln(N_Z)]^{1/2}$$

(15)

in which \( N_Z \) now represents the average number of cycles of the combined response \( Z(t) \) during the time interval \( T \).

It only remains to estimate \( N_Z \). For this purpose, it is convenient to note that for a standard Gaussian process such as \( Z(t) \), the zero-crossing rate \( \nu_0 \) is proportional to the rms \( \sigma_Z \) of the derivative process:

$$\alpha_Z = \omega Z_0 \sigma_Z = 2\pi \nu_0 \sigma_Z = 2\pi \nu_0$$

(16)

Here \( \omega Z_0 \) is the mean frequency in radians/sec and the factor of \( 2\pi \) gives \( \nu_0 \) in Hz. The last equality is based on the fact that \( \sigma_Z = 1 \).
To use (16), we differentiate (13):
\[
\dot{Z}(t) = \dot{U}_1(t)\cos(\theta) + \dot{U}_2(t)\sin(\theta)
\]  
(17)
The variances then satisfy
\[
\sigma^2_Z = \sigma^2_{\dot{U}_1}\cos^2(\theta) + \sigma^2_{\dot{U}_2}\sin^2(\theta)
\]  
(18)
and (16) and (18) give
\[
N_Z = [N_1^2\cos^2(\theta) + N_2^2\sin^2(\theta)]^{1/2}
\]  
(19)
in which \( N_1 \) is the average number of cycles of \( U_1(t) \) during the time interval \( T \). In other words, the average number of cycles \( N_Z \) of the combined demand \( Z(t) \) is found from an SRSS-like operation on \( N_1 \) and \( N_2 \), weighted by \( \cos^2(\theta) \) and \( \sin^2(\theta) \), respectively.

We can now summarize results—particularly (15) and (19)—into a recipe to generate a \((u_1,u_2)\) contour. Similar to the parametric form in (12)-(13) we have
\[
N_Z(\theta) = [N_1^2\cos^2(\theta) + N_2^2\sin^2(\theta)]^{1/2}; \theta = [0,2\pi]
\]  
(20)
\[
\beta(\theta) = \left(2\ln[N_Z(\theta)]\right)^{1/2}
\]  
(21)
\[
(u_1,u_2) = \beta(\theta)[\cos(\theta),\sin(\theta)]
\]  
(22)
While this specification may seem a bit complex, it is readily implemented by sweeping the angle \( \theta \) around a circle and plotting the (angle-dependent) radius \( \beta(\theta) \). It is clear that the limiting values \( \beta_1 = [2\ln(N_1)]^{1/2} \) and \( \beta_2 = [2\ln(N_2)]^{1/2} \) are found when \( \theta = 0 \) and \( \pi/2 \), respectively.

There are at least two possible ways to use the dynamic FORM contour: 1) as stated earlier, the true FORM solution is to find the point \( u^* \) on the contour such that the vector \( u^* \) is orthogonal to the design limit passing through this point, and 2) a cruder, but conservative, approximation is to find the largest demand value anywhere on the contour. Note that these two approaches are the same for the static problem in which the contours are circular (or spherical in \( n \)-dimensional space), but they generally give different results in dynamic problems. This will be investigated in the numerical examples.

3 Numerical Example

For a ship example, Leira [8] considered a combined vertical midspan bending moment response, \( Y(t) \), of the form
\[
Y(t) = X_1(t) + X_2(t)
\]  
(23)
in which \( X_1(t) \) is the still-water bending moment, and \( X_2(t) \) is the wave-induced bending moment. It is further assumed that \( X_1 \) and \( X_2 \) have average cycle numbers of \( N_1 = 120 \) and \( N_2 = 10,000 \) over the duration of interest. Assuming each \( X_i(t) \) is a translation process of a standard normal process \( U_i(t) \), \( N_1 \) and \( N_2 \) also give the average number of cycles of the underlying Gaussian processes \( U_1(t) \) and \( U_2(t) \). Note that the \( N_i \) cycle count must be the rate of upcrossings of the median of \( X_i(t) \), since the median of \( X_1 \) maps into zero for \( U_1 \).

Figure 3 compares the \((u_1,u_2)\) contours generated in this case according to: 1) the ellipse of Leira (14), and 2) the FORM-based contours of (20)-(22). Note that this figure, and all others that plot \( u_1 \) versus \( u_2 \), have been forced to be square in aspect ratio, so that angles are visually accurate. The contours are only plotted in the first quadrant, since that is the area that can give large values of \( Y \).

In the limiting cases when either \( U_1 \) or \( U_2 \) goes to zero, the process has the characteristic extreme given by (12):
\[
\beta_1 = [2\ln(120)]^{1/2} = 3.09
\]
\[
\beta_2 = [2\ln(10000)]^{1/2} = 4.29
\]  
(24)
Both contours in Fig. 3 show these values for the limiting angles of \( \theta = 0 \) and \( \theta = \pi/2 \). For intermediate angles, however, the FORM-based contour consistently yields larger values of the radius \( \beta \). The differences are especially pronounced when the "slow" process \( U_1 \) dominates. When a small contribution of the "fast" process \( U_2 \) is introduced (i.e., \( \theta \) slightly greater than 0) the cycle number \( N_Z \) of the combined demand rises quickly, due to the SRSS operation in (19). This effect appears not to be reflected in the ellipsoidal contour.

The plot also highlights the difference mentioned earlier between the two methods of using the FORM-based contour. While the true (FORM-predicted)
demand lies somewhere on this contour, it may or may not be the maximum demand anywhere on the contour. For example, if we have a limit state of \( u_1 = \text{constant} \) then maximum demand anywhere on the contour lies at the rightmost point on the contour; here, a value of approximately 3.8 in Fig. 3. Meanwhile, the orthogonality of \( \mathbf{u}^* \) and the limit state gives \( u_2 = 0 \) (\( \theta = 0 \)) and a FORM demand of 

\[ y = y_1 = \beta_1(2\ln(N_1))^{1/2} = 3.09 \] 

Thus, this is a situation for which the capacity requirement is significantly different for the two methods. For greater accuracy, the FORM-based contour should not be searched over its entirety, but rather evaluated at a specific angle \( \theta \) relevant to the problem at hand. In many practical cases, however, the maximum demand on the FORM contour is a close approximation of the true FORM design point. This is shown in the examples that follow.

**Case 1: Base Case**

Note that the U-space contours in Fig. 3 depend only on the number of cycles, \( N_1 \) and \( N_2 \), shown by \( U_1 \) and \( U_2 \), respectively, over the duration of interest. We will retain the values of \( N_1 = 120 \) and \( N_2 = 10,000 \) through all the examples in this section; thus, the contours in U-space will remain constant throughout.

To find the \( h_1(u_i) \) functions that will transfer these contour to the physical X-space we must know the probability distributions of \( X_1 \) and \( X_2 \). Following Leira, we use the following mean and standard deviation values:

- Moments of \( X_1 \): \( m_1 = 1,000 \quad \sigma_1 = 500 \) 
- Moments of \( X_2 \): \( m_2 = 500 \quad \sigma_2 = 500 \) (25)

Also, we will consider \( X_1(t) \) and \( X_2(t) \) to be independent processes.

The original Leira example models \( X_1 \) as Gaussian and \( X_2 \) as being exponentially distributed. We will return to this case later in the paper, but first we consider the simpler case in which both \( X_1 \) and \( X_2 \) are Gaussian. In this case \( Y \) is also Gaussian, so that exact analytical results are available. These should agree with the FORM-based contour results, as they too are consistent with a linear combination of Gaussian demand terms.

Because \( Y(t) = X_1(t) + X_2(t) \) is Gaussian, its exact characteristic value is given by a result analogous to (12), but now shifted and rescaled to reflect the mean \( m_Y \) and standard deviation \( \sigma_Y \) of \( Y \):

\[ y_{cr} = m_Y + \sigma_Y[\ln(N_Y)]^{1/2} \] (26)

and \( N_Y \) is given as in (16)-(19) as

\[ N_Y = \nu_Y(0)T = \frac{\sigma_Y T}{(2\pi)^{1/2}} = \frac{T[\omega_1^2 \sigma_1^2 + \omega_2^2 \sigma_2^2]^{1/2}}{(2\pi \sigma_Y)} \] (27)

Using the data in (25) gives the necessary statistics for \( Y \) as

\[ m_Y = m_1 + m_2 = 1500 \]

\[ \sigma_Y = [\sigma_1^2 + \sigma_2^2]^{1/2} = 707 \]

\[ N_Y = [N_1 \sigma_1^2 + N_2 \sigma_2^2]^{1/2} / \sigma_Y = 7070 \] (28)

Substituting these values into (26) gives the exact result as

\[ y_{cr} = 1500 + 707[\ln(7072)]^{1/2} = 4480 \] (29)

To use contours in this case, we may consider results either: 1) in the standard Gaussian U space, or 2) in the original space of the Gaussian physical X variables. In approach 1), we can retain the contours from Fig. 3; however, we need to restate the limit state in terms of \( u_1 \) and \( u_2 \):

\[ h_1(u_1) + h_2(u_2) = y \] (30)

in which

\[ h_i(u_i) = m_i + \sigma_i u_i \] (31)

giving the limit state as

\[ \sigma_1 u_1 + \sigma_2 u_2 = y - m_1 - m_2; u_1 + u_2 = (y - 1500) / 500 \] (32)

This approach is shown in Fig. 4. The contours in this figure are identical to those in Fig. 3. The form of (32) shows that the \((u_1, u_2)\) limit state is a straight line with an angle of \(-45^\circ\). Thus, the orthogonal vector to the critical point \( \mathbf{u}^\ast \) has an angle of \( \theta^\ast = +45^\circ \). Using \( \beta = [2\ln(N_Y)]^{1/2} \) gives \( u_1^\ast = u_2^\ast = 0.5^{1/2} \beta = 2.98 \) and (32) gives the value of \( y_{cr} = 4480 \). As expected, the result is exact. Figure 3 also shows that in this case, the critical point is also the maximum value of \( y \) on the circular contour with radius \( \beta = [2\ln(N_Y)]^{1/2} = 4.21 \).

In contrast, the elliptical contour is shown to be non-conservative, yielding the estimate of \( y_{cr} = 4150 \). To put this difference into better perspective, it is convenient to focus on the dynamic response by removing the deterministic portion of the response (i.e., the \( m_Y \) term). This gives values of \( y_{cr} - m_Y \) of 2980 and 2650 for the exact result and the elliptical approximation, respectively. As Table 1 notes, this is a response underestimation of 11.1%.

Figure 5 shows the same results in the form of
(x₁,x₂) contours. The contours from Fig. 3 have been transformed to X-space by using (31). This permits the y = x₁ + x₂ limit state to be plotted directly in the units of the physical problem. Of course, the numerical results are the same as in Fig. 4.

**Case 2: Dominant Still-Water Moment**

We now modify the statistics in (25) so that the still-water bending moment, X₁(t), has greater effect. Specifically, we rescale X₁(t) by a factor of 3, and X₂(t) by a factor of 0.5:

- Moments of X₁: m₁ = 3000, σ₁ = 1500
- Moments of X₂: m₂ = 250, σ₂ = 250

This set of parameter values has also been studied by Leira. Although he shows results only for exponentially distributed X₂, we, for simplicity, again start here with both X₁ and X₂ being Gaussian.

Using the moments in (33), we repeat calculations analogous to (28)-(29):

\[
\begin{align*}
    m_Y &= m_1 + m_2 = 3250 \\
    \sigma_Y &= \sqrt{\sigma_1^2 + \sigma_2^2} = 1520 \\
    N_Y &= \left[ N_1^2 \sigma_1^2 + N_2^2 \sigma_2^2 \right]^{1/2} / \sigma_Y = 1650 \\
    y_{cr} &= 3250 + 1520 \ln(1650) \approx 9100
\end{align*}
\]

The limit state in U-space is again given by (32):

\[\sigma_1 u_1 + \sigma_2 u_2 = y - m_1 - m_2.\]

Note that the slope of this line is \(u_2 / u_1 = -\sigma_1 / \sigma_2 = -6\), and the slope of the orthogonal \(u^*\) is 1/6.

Figure 6 compares the U-space contours with the limit state \(y = g_1(u_1) + g_2(u_2)\) for various y values. Because these are in U-space, these contours are the same as in Fig. 4 for the base case problem. The different statistics here [(33) versus (25)] are reflected in the different U-space limit states in the two figures. Compared with Fig. 4, constant response lines in Fig. 6 are more nearly vertical, reflecting the increased statistical importance of X₁ in this case.

As before, the elliptical contour is non-conservative, yielding \(y_{1} = 8050\) compared to the exact result of \(y_{cr} = 9100\). This exact result again is contained on the FORM-based contour for \(\beta = [2 \ln(N_Y)]^{1/2} = 3.85\). Using the same logic as in the preceding section gives the angle of the vector to the critical point \(u^*\) as \(\theta^* = \tan^{-1}(\sigma_1 / \sigma_2) = 9.46^\circ\). Incidentally, this formula of \(\theta^* = \tan^{-1}(\sigma_1 / \sigma_2)\) also applies to the simpler situation in Case 1, where the angle was obvious. The line with slope-angle of 9.46° is also shown in Fig. 6. Note also that if we instead simply evaluate the worst demand anywhere on the FORM contour, we find the only slightly larger value of \(y_{max} = 9300\). Table 2 summarizes these values and their relative errors. Figure 7 shows the same results in X-space. Since both X₁ and X₂ are Gaussian, the

---

Table 1. Comparison of contour results, base case

<table>
<thead>
<tr>
<th></th>
<th>Total</th>
<th>Mean Removed</th>
<th>Percent Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>4480</td>
<td>2980</td>
<td>...</td>
</tr>
<tr>
<td>Maximum over FORM contour</td>
<td>4480</td>
<td>2980</td>
<td>none</td>
</tr>
<tr>
<td>Maximum over Elliptical Contour</td>
<td>4150</td>
<td>2650</td>
<td>-11.1%</td>
</tr>
</tbody>
</table>
transformation from Fig. 6 to Fig. 7 is merely a shifting and rescaling.

**Cases 3 and 4: Exponential $X_2$**

Finally, we reconsider cases 1 and 2 under the alternate condition, as suggested by Leira, of $X_1$ being Gaussian and $X_2$ being exponentially distributed. We return first to the base case statistics in (25). Figures 8 and 9 show the resulting contours (and limit states) in $U$-space and $X$-space, respectively. The transformation between $x_2$ and $u_2$ is now given by

$$u_2 = f_2(x_2) = \Phi^{-1}(1 - e^{-x_2/u_2})$$

$$x_2 = h_2(u_2) = -\sigma_2 \ln[1 - \Phi(u_2)]$$

(35) to calculate the slope of the contour at any point. In particular, we can note that an incremental movement in $X$-space with slope $\delta x_2 / \delta x_1$ corresponds to a movement in $U$-space with slope

$$\frac{\delta u_2}{\delta u_1} = \frac{f_2'(x_2)\delta x_2}{f_1'(x_1)\delta x_1} = \frac{f_2'[h_2(u_2)]\delta x_2}{f_1'[h_1(u_1)]\delta x_1} = \frac{h_1'(u_1)\delta x_2}{h_2'(u_2)\delta x_1}$$

(36)

A constant-$y$ line in $X$-space has $\delta x_2 = -\delta x_1$ so that the slope of the $U$-space contour at point $u^*$ must be $-h_1'(u_1^*) / h_2'(u_2^*)$. Thus, the orthogonal vector $u^*$ must have a slope of $h_2'(u_2^*) / h_1'(u_1^*)$, and the numerical condition for the critical point can be written as

$$\frac{u_2^*}{u_1^*} = \frac{h_2'(u_2^*)}{h_1'(u_1^*)}$$

(37)

It is found that the point on the contour satisfying this relationship has $\theta^* = 77.4^\circ$ and $\beta^* = 4.29$, giving $u^* = (0.935, 4.48)$. The mapping into $X$-space gives $x^* = (1470, 5570)$ so that $y_{cr} = 7040$.

In comparing the new numerical results in Figs. 8 and 9 with those of case 1 in Figs. 4 and 5, the most striking feature now is the strong dominance of $X_2$, the wave-induced moment. This is due to the relatively broad tail of the exponential distribution, as compared to the Gaussian distribution of $X_1$.
to that of a normal distribution with the same first two moments.

This situation—where the “fast” process dominates—is one where the Leira and FORM contours show the greatest agreement. Thus, the FORM contour value of $y_{cr} = 7040$ is just slightly above the value of $y_{cr} = 6950$ from the ellipsoidal contour. Also, as in case 1, the maximum $y$ value on the FORM contour almost coincides with the FORM design point.

Lastly, we return to the dominant still-water statistics of (33), now with $X_2$ exponentially distributed. Because $X_2$ is of relatively little importance here, we may anticipate similar results to our earlier results for case 2 (Figs. 6, 7). Specifically, we may anticipate 1) FORM-based results for $y_{cr}$ on the order of 9000; 2) Ellipse-based $y_{cr}$ results on the order of 8000; and 3) the FORM design point lying slightly below the maximum value on the FORM contour.

Figs. 10 and 11 confirm these expectations. Specifically, the maximum value on the FORM contour is $y_{cr} = 9350$. This is slightly above the design point value, found to be $y_{cr} = 9180$. Again, this value has the property of a minimum distance point; i.e., the response surface is locally perpendicular to a line from the origin, now with slope of about 0.25. In contrast, the ellipsoidal contour yields $y_{cr} = 8100$ (the
value also reported by Leira). While the exact result is not available for comparison, we may expect in this case that the change from normal to exponential \( X_2 \) has only a slight effect in this case where \( X_1 \) dominates. Hence, as in Table 2, we would expect the FORM result to remain quite accurate in this case.

4 Non-FORM Alternatives

Two calculations that do not use FORM approximations will now be presented. The first of these differs from the FORM approach only by evaluating crossings of the true nonlinear limit state, rather than a linear approximation in U-space. This procedure generally involves numerical integration, and is quite inefficient compared to FORM. It is introduced here only to investigate the accuracy of the FORM approximation for a system of the type investigated in Sec. 3. The second alternative approach is to replace the translation-process assumption of independence of \( U(t) \) and \( \dot{U}(t) \) with an assumption of independence of \( X(t) \) and \( \dot{X}(t) \).

It is important to notice that \( X(t) \) and \( \dot{X}(t) \) are generally not independent when \( X(t) \) is a translation process. In particular, \( \dot{X}_i(t) = h_i[U_i(t)]\dot{U}_i(t) \), which shows that \( \dot{X}(t) \) generally depends on both of the independent quantities \( U(t) \) and \( \dot{U}(t) \), while \( X(t) = h[U(t)] \) depends only on one of the two. An almost trivial exception is when all components of \( X \) are Gaussian, since then the term \( h_i[U_i(t)] \) is a constant for each \( i \) value so that \( \dot{X}(t) \) does not depend on \( U(t) \).

The problems investigated here, as in Sec. 3, have a demand function of \( Y(t) = X_1(t) + X_2(t) \), but the number of crossings is across the true line \( Y = y \) in \( X \)-space, rather than the form approximation of a straight line in \( U \)-space. As before, \( X_1(t) \) and \( X_2(t) \) are considered to be independent processes. The exact rate of crossings (based on the Poisson approximation) of any level \( y \) can be written directly as a triple integral in \((X,X)\)-space. This triple integral can be separated into a double integral over the space of \( \dot{x}_1 + \dot{x}_2 > 0 \) then an integral over the entire line of \( x_1 + x_2 = y \):

\[
I(y,x_1) = \int_{-\infty}^{\infty} \int_{x_1}^{\infty} (\dot{x}_1 + \dot{x}_2) p_{X_1}(\dot{x}_1 | X_1 = x_1) \, d\dot{x}_2 d\dot{x}_1
\]

\[
Y(y) = \int_{-\infty}^{\infty} I(y,x_1) p_{X_1}(x_1) p_{X_2}(y-x_1) \, dx_1 \tag{38}
\]

One way to simplify this rather cumbersome result is to use the Rosenblatt mapping into \( U \)-space and the resulting condition that \( U_1(t) \) and \( U_2(t) \) are independent Gaussian processes. This allows simplification to a single integral of

\[
N_Y(y) = \frac{1}{(2\pi)^{1/2}} \int e^{-(u_1^2 + u_2^2)/2} h_2'(u_2) \left( N_1^2 [h_1'(u_1)]^2 + N_2^2 [h_2'(u_2)]^2 \right)^{1/2} \, du_1 \tag{39}
\]

in which

\[
u_2 = f_2[y - h_1(u_1)] \tag{40}
\]

and with the \( f_i(x_i) \) and \( h_i(u_i) \) functions coming from the Rosenblatt mapping, as in (3). Numerical integration is generally required for evaluation of this final integral.

There are important special cases in which we may be interested in non-Gaussian processes that are not translation processes. That is, there are situations in which the Rosenblatt mapping does not give \( U(t) \) and \( \dot{U}(t) \) as independent processes, so none of the results presented so far in this paper are appropriate. For example, there is a non-trivial class of dynamic models for which the response gives the random variable \( \dot{X}(t) \) as being Gaussian and independent of the random variable \( X(t) \) at the same instant of time. In particular, this occurs if a Gaussian white noise excitation is applied to a vibratory mechanical or structural system with nonlinear stiffness and linear damping. Of course this model is idealized, but it has often been used to approximate physical problems. We will give some limited results for this class of problems with independent \( X(t) \) and \( \dot{X}(t) \).

Again, we will limit our attention here to the special case in which the demand is \( Y(t) = X_1(t) + X_2(t) \). In addition, we say that \([\dot{X}_1(t), \dot{X}_2(t)]\) is a pair of Gaussian random variables, and \([X_1(t), \dot{X}_1(t), X_2(t), \dot{X}_2(t)]\) is a set of independent random variables for any given value of \( t \). The variance expression for \( \dot{Y}(t) \) can be written as

\[
\sigma_{\dot{Y}}^2 = \sigma_{\dot{X}_1}^2 + \sigma_{\dot{X}_2}^2 = \frac{2\pi}{T} \left[ \frac{N_1^2}{p_{X_1}(x_{1m})} + \frac{N_2^2}{p_{X_2}(x_{2m})} \right] \tag{41}
\]

in which the procedure of (16) has been used to write the \( \sigma_{X_i} \) standard deviation values in terms of crossings during an interval \( T \). The term \( x_{im} \) in (41) denotes the median value of \( X_i(t) \) so that \( N_i \) is defined in the same way as in the FORM analysis. For the present calculation, though, this is an arbitrary choice. The ratio \( N_i / p_{X_i}(x_{im}) \) in (41) could equally
well be replaced by $\nu_{X_i}(\tilde{x}_i)T/p_{X_i}(\tilde{x}_i)$ for any value of $\tilde{x}_i$. Using (41) in the standard expression for the number of crossings by $Y(t)$ gives

$$
N_Y(y) = TV_Y(y) = T p_Y(y) \frac{\sigma_Y}{(2\pi)^{1/2}}
$$

$$
= (2\pi)^{1/2} p_Y(y) \left( \frac{N_1^2}{p_{X_1}(x_{1\text{m}})} + \frac{N_2^2}{p_{X_2}(x_{2\text{m}})} \right)^{1/2} \tag{42}
$$

As before, the design point is chosen to have $N_Y(y) = 1$, so that one can find this point by searching the expression in (42) over a sequence of $y$ values. The only remaining difficulty is in finding the pdf of $Y$. This is generally quite simple for the situation considered here with only two $X_i(t)$ components. In particular, one can always use numerical integration of

$$
p_Y(y) = \int p_{X_1}(x_1)p_{X_2}(y-x_1) \, dx_1 \tag{43}
$$

if an analytical solution is not available. Another alternative is to use the Rosenblatt mapping to replace this integral with one in U-space. Although this introduction of the Gaussian distribution may seem desirable, the $(y-x_1)$ argument maps into the nonlinear form of $f_2[y-h_1(u_1)]$, so that there does not seem to be any real simplification of the integral.

For the special case in which $X_1$ is Gaussian and $X_2$ is exponential, as in Sec. 3, it is relatively easy to analytically evaluate the integral in (43) giving

$$
p_Y(y) = \frac{1}{\sigma_2} \exp \left( \frac{(2y-\sigma_1^2/\sigma_2^2)}{2\sigma_2^2} \right) \Phi \left( \frac{(y-\sigma_1^2/\sigma_2^2)}{2\sigma_1^2} \right) \tag{44}
$$

Although this expression is uncomplicated, its use is not always trivial. For situations with $\sigma_2 << \sigma_1$ one may need to investigate $y$ values that give such a negative argument of the $\Phi$ function that either very high precision or an asymptotic expansion is necessary to achieve proper numerical results.

5 Extended Numerical Comparisons

We have made additional numerical calculations in an attempt to identify situations in which any of the calculation techniques may give questionable values of the design point. These calculations have been limited to the situation of the demand being $Y = X_1 + X_2$ with Gaussian- $X_1$ and exponential- $X_2$, but the range of parameters is much broader than was considered for the ship-bending problem. In particular, $\sigma_2/\sigma_1$ was varied from $10^{-3}$ to $10^3$ and $N_2/N_1$ varied from $10^{-4}$ to $10^4$. The values of both $N_1$ and $N_2$ were restricted to be no smaller than 100. The focus was on situations in which $N_i > N_j$ when $\sigma_i < \sigma_j$, since these are situations in which the component with small variance can still have a significant effect on $p_f$, since it contributes significantly to the crossing rate for $Y$.

The first issue studied relates to the accuracy of the inverse FORM method. Over the very broad range of situations it was found that the FORM results were in very good agreement with the numerical integration results of (39). The discrepancy was usually less than 1%, although there were limited situations for which it was slightly larger. The largest discrepancy observed was about 4%, and occurred for the extreme case of $\sigma_1 = \sigma_2$, $N_2 = 1000 N_1$. Over all, it seems that the inverse FORM method gives very accurate results.

The next numerical comparison was between the full FORM method and the simpler choice of seeking the largest demand anywhere on the $p_f$-contour. Recall that there was little difference between these methods for the ship-bending problem. It was found, though, that the difference could be much larger in situations where one of the components had a very small value of $\sigma$ and a very large value of $N$. For $\sigma_2 = 10^{-3} \sigma_1$, $N_2 = 10^5 N_1$ the maximum-$Y$ approach gave the design point about 38% larger than the FORM result. Similarly, $\sigma_1 = 10^{-3} \sigma_2$, $N_1 = 10^3 N_2$ gave a 76% discrepancy. Of course, the maximum-$Y$ approach is always conservative. Tables 3 and 4 give some of the numerical data.

<table>
<thead>
<tr>
<th>$N_1=100$</th>
<th>$Y_{\text{min}}$ and $Y_{\text{FORM}}$ for $N_i=100$, $N_2&gt;N_i$, $\sigma_i&lt;\sigma_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_f/\sigma_i$</td>
<td>$N_2/N_1=10$</td>
</tr>
<tr>
<td>1</td>
<td>1.00</td>
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<td>0.01</td>
<td>1.07</td>
</tr>
<tr>
<td>0.001</td>
<td>1.07</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$N_2=100$</th>
<th>$Y_{\text{min}}/Y_{\text{FORM}}$ Design Point Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_f/\sigma_i$</td>
<td>$N_2/N_1=0.1$</td>
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<tr>
<td>1</td>
<td>1.03</td>
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<tr>
<td>10</td>
<td>1.11</td>
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<tr>
<td>100</td>
<td>1.11</td>
</tr>
<tr>
<td>1000</td>
<td>1.10</td>
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</tbody>
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The final comparison is between the FORM (or numerical integration) result for the translation-process model and the result for the model in which $\tilde{x}(t)$ is Gaussian and independent of $X(t)$. It was found that the results from the two models were often in very good agreement, but the independent-$\tilde{x}$ model sometimes gave smaller values of the design point.
The largest difference observed was about 20\%, and it occurred when $\sigma_2 \geq 100\sigma_1$, $N_2 \leq 0.1\sigma_1$. A similar, but smaller, difference of about 15\% was observed for $\sigma_2 = \sigma_1$, $N_2 > N_1$. Tables 5 and 6 give some of the numerical data. It should be noted that the differences in these calculations do not reflect an error in either model, but rather the effect of model choice. The translation model sometimes gives somewhat larger values of the design point than does the independent $X$ model. Thus the usual FORM approach with the translation model is conservative in the situation in which $X(t)$ is actually independent of $X(t)$.

Table 5. Comparison of translation model (TM) and independent derivative model (IDM) for $N_1=100$, $N_2>N_1$, $\sigma_2<\sigma_1$

<table>
<thead>
<tr>
<th>$\sigma_2/\sigma_1$</th>
<th>$N_2/N_1=10$</th>
<th>$N_2/N_1=100$</th>
<th>$N_2/N_1=1000$</th>
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<td>1.000</td>
<td>1.000</td>
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</tr>
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</table>

Table 6. Comparison of translation model (TM) and independent derivative model (IDM) for $N_2=100$, $N_2<N_1$

<table>
<thead>
<tr>
<th>$\sigma_2/\sigma_1$</th>
<th>$N_2/N_1=0.1$</th>
<th>$N_2/N_1=0.01$</th>
<th>$N_2/N_1=0.001$</th>
<th>$N_2/N_1=0.0001$</th>
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</tr>
<tr>
<td>10</td>
<td>0.852</td>
<td>0.992</td>
<td>1.000</td>
<td>1.000</td>
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<tr>
<td>100</td>
<td>0.790</td>
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<td>1000</td>
<td>0.789</td>
<td>0.790</td>
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</table>

6 Summary and Conclusions

The inverse FORM method, previously used for static problems, has been extended to apply to dynamic situations. The method is based on the use of probability-of-failure contours to identify design points. The full FORM approach is not based on choosing the maximum-demand point on the FORM contour, although that is a simple alternative. It has been shown that results of the new technique agree very well with those of inefficient numerical integration. For a specific problem of bending moment in a ship the FORM results agree well with the simpler maximum-demand approach, but it has been shown that that maximum-demand is overly conservative in some more extreme situations.

A brief investigation has been made of an alternative to the translation-process assumption that is usually used in FORM. This alternative model treats $\dot{X}(t)$ as Gaussian and independent of $X(T)$. Numerical results have shown that this independent-derivative model sometimes gives significantly smaller demand values than inverse FORM, so that use of inverse FORM is conservative in situations in which the other model is more accurate.

References