DESIGN CONTOURS FOR LOAD COMBINATIONS: GENERALIZING INVERSE FORM METHODS TO DYNAMIC PROBLEMS

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In static structural reliability problems involving a set of random variables, it is common to relate the failure probability to the event that a safety margin becomes negative. For dynamic responses we commonly seek the mean out-crossing rate into the unsafe region. Standard FORM (First-Order Reliability) methods can be applied to both the static and dynamic problems. A challenge comes in the design context, when a target probability of failure or crossing rate is specified and a particular design variable (e.g., capacity) is sought. In the static case, inverse FORM methods have been suggested. The resulting "design contour" may be viewed, in the transformed space of standard Gaussian variables, as the locus of all possible FORM design points that can give the desired probability of failure value. These contours have been previously used for the static problem to determine "design contours" for various problems. We seek here to develop analogous design contours for the dynamic problem, based on FORM principles, again defining the contour as a locus of all possible FORM design points. We apply these contours to study a practical load combination problem involving the midspan vertical bending moment of a ship. Finally, we present some data on the accuracy of several methods of analysis, and the effect of modeling choices for the input variables.

Keywords: Reliability, FORM, dynamic, design contours.

1 Introduction

In structural reliability, it is common to relate the failure probability $p_f$ to a limit state (or failure surface) $g(X)$ involving a vector $X$ of random variables. The probability distribution of $X$ is presumed to be known and failure is presumed to occur if $g(X) < 0$. In many practical situations the random quantities of interest relate to the demand $d$ on the structure, while the capacity $c$ is determined by the designer. In this situation $c$ is treated as deterministic and the limit state can be written as $g(X) = c - d(X)$. This study will be written using this simplified formulation. In cases of dynamic responses to a load process $X(t)$, we commonly use a Poisson model to estimate the probability of failure, $p_f(T')$, during a specific time interval $T$. This only requires knowledge of the mean out-crossing rate by $X(t)$ into the unsafe region, denoted as $\nu_d|X(t)|\{c\}$.

Standard FORM (First-Order Reliability) methods can be used to approximate $p_f$ for given probability distributions of $X$ for either the static or dynamic situation (e.g., Madsen et al., 1986). First the Rosenblatt transformation is used to map $X$ into a vector $U$ of independent standard Gaussian random components. This maps the failure condition...
of \( d(X) = c \) into a corresponding limit state \( d(U) = c \). In either the static or the dynamic problem, interest is focused on the most likely point on this limit state. In the standard Gaussian U-space this point \( u^* \) is simply the point on the limit state that is nearest the origin. The FORM analysis then approximates the limit state by its tangent plane at the point \( u^* \), thus simplifying the calculation of the probability of failure. In the static case the FORM approximation of \( p_f \) depends only on the distance of \( u^* \) from the origin, denoted as \( \beta(u^*) \), and in the dynamic case it depends on crossing rate information.

The design problem is the inverse of the above procedure. That is, there is now a given value of \( p_f \) but at least some aspects of the capacity \( c \) are unknown, since they depend on design parameters that have not yet been chosen. In the static case, inverse FORM methods have been suggested (e.g., Winterstein et al., 1993, Der Kiureghian et al., 1994). The resulting "design contours" may be viewed as the locus, in U space, of all possible design points that produce the desired \( p_f \). These contours are simply spheres in U-space for the static problem, since \( p_f \) for any point \( u \) depends only on the distance from the origin, \( \beta(u) \). These contours have been used by Winterstein et al. (1999), Forristall and Cooper (1997), Saranyssoontorn and Manuel (2004), and Havre and Winterstein (2009) to derive "design contours" for various problems.

We seek here to extend the inverse FORM method to develop analogous contours for dynamic problems. One prior proposal for this situation uses elliptical contours in U space (Leira 2012). We develop alternative contours by using FORM principles to identify the locus of all possible design points having the crossing rate that gives the specified probability of failure. Since the FORM estimate of the crossing rate depends not only on \( \beta(u) \) but also on given information about the crossing rates of \( X \), the U-space contours are generally not spherical. We apply these contours to study a load combination problem involving the midspan vertical bending moment of a ship. Finally, we introduce two non-FORM models and give some comparison of numerical results.

2 Mathematical Formulation

The notation used here is based on using capital letters for random variables or processes and lower case letters for deterministic quantities, or "dummy values" of these random terms.

The equations for the probability of failure, as described above, are

Static case: \[ p_f = P[d(X) > c] \] (1)

Dynamic case:

\[ p_f = 1 - \exp(-T \cdot v_{d[X]}(c)) \] (2)

in which the dynamic result is based on the Poisson approximation for the probability of a crossing by the stationary time history \( X(t) \) into the failure region during a time interval of duration \( T \).

We will write the Rosenblatt transformation and its inverse as

\[ U = f(X) \quad X = h(U) \] (3)

This mapping of \( X \) into the standard Gaussian \( U \), also maps the limit state \( d(X) = c \) into \( d(u) = d[h(u)] = c \). The design point (or "characteristic point") \( u* \) is the most likely point on the limit state, which means it gives the minimum value of \( \beta(u) = u \) for all \( u \) with \( d(u) = c \).

The FORM approximation then replaces the limit state in U-space with the plane that is tangent to the limit-state surface at \( u^* \). With this approximation, the failure condition of \( d(u) > c \) can be written as \( Z > \beta(u^*) \) in which \( Z \) is a unit-variance scalar Gaussian random variable that is a linear function of \( U \). One way
to think of \( Z \) is as the projection of \( U \) onto the line from the origin to \( u^* \). For this standard Gaussian measure of demand, Eq. (1) gives

\[
p_f = \Phi[-\beta(u^*)] \tag{4}
\]

in which \( \Phi \) denotes the cumulative distribution of the standard Gaussian random variable. Similarly the dynamic probability of failure is based on the rate of crossing of the FORM plane by \( U(t) \). This is the same as crossings by the unit-variance Gaussian time history \( Z(t) = Z[U(t)] \). Thus

**Dynamic case:**

\[
p_f = 1 - \exp(-\nu Z(0) T e^{-\mu^2(u^*)/2}) \tag{5}
\]

Finding the crossing rate of \( Z(t) \) in the Gaussian \( U \)-space requires knowledge of the joint distribution of \( Z(t) \) and \( \dot{Z}(t) \). Some caution is needed at this point. The obvious choice is to say that \( Z(t) \) is a Gaussian process, for which \( Z(t) \) and \( \dot{Z}(t) \) are jointly Gaussian. We will use this assumption in the following FORM analysis, but it is not directly implied by the Rosenblatt transformation in Eq. (3). All that is assured by the Rosenblatt transformation is that \( U(t) \) is a standard Gaussian random variable for any value of \( t \). It does not assure that \( U(t_1) \) and \( U(t_2) \) are jointly Gaussian for \( t_1 \neq t_2 \), which is required in order to have \( U(t) \) be Gaussian, and jointly Gaussian with \( U(t) \)—that is to have \( U(t) \) be a standard Gaussian process. The special case in which \( U(t) \) is a standard Gaussian process requires that \( X(t) \) is a translation process (Grigoriu, 1995), which means that Eq. (3) gives a standard Gaussian process \( U(t) \).

For a standard Gaussian process we know that

\[
\nu(u) = \nu_0 e^{-u^2/2} \tag{6}
\]

in which \( \nu_0 \) is the mean crossing rate of zero. Thus, using the translation process assumption in Eq. (5) gives

**Dynamic case:**

\[
p_f = 1 - \exp(-\nu Z(0) T e^{-\mu^2(u^*)/2}) \tag{7}
\]

The design problem is the inverse of the above procedure. That is, there is now a given value of \( p_f \) and the problem is to find the design point \( x^* \) out of all the \( x \) values that give this value of \( p_f \). We will consider \( p_f \) contours in \( X \)-space or \( U \)-space that contain all points with the specified value of \( p_f \). The problem is to find the point \( u^* \) that not only is on the contour but also is the most likely point on a limit-state surface passing through that point.

Since the value of the capacity \( c \) is not known \textit{a priori} in the design problem, neither is the limit state. Thus, one can picture a multi-dimensional limit-state map in either \( X \)-space or \( U \)-space with each limit-state surface corresponding to a different value of \( c \). One such contour surface will pass through each \( u \) point. Since \( u^* \) is the most likely point on the appropriate limit state, it is necessary that the limit-state surface is locally orthogonal to the vector from the origin to \( u^* \). This orthogonality condition identifies \( u^* \), and this point is then mapped back into the \( X \)-space to give the so-called "design point" as \( x^* = h(u^*) \) and the design value of the capacity \( c \).

### 2.1 Static inverse FORM

The demand quantity in the inverse problem may be of various forms, such as a load, a load effect, or a structural response. The object is to find a design value of this quantity based on a given probability of failure \( p_f \).
For simplicity we will consider a case in which $X$ and $U$ have only two dimensions. Then any point $u$ can be written in a polar notation form of

$$ u = (u_1, u_2) = \beta(u)[\cos(\theta), \sin(\theta)] $$

(8)

in which $\theta = \tan^{-1}(u_2 / u_1)$. From Eq. (4) we then have the constant-$p_f$ contour as the set of $u$ values satisfying

$$ \beta(u) = -\Phi^{-1}(p_f) $$

(9)

and Eqs. (8) and (9) give a parametric representation of the contour by considering $\theta = [0, 2\pi]$. Obviously, this contour will be circular in this two-dimensional space.

One can now envision a family of limit state curves in the $(u_1, u_2)$ plane. Each of these curves represents a collection of points satisfying $c = d(u_1, u_2)$ for a particular value of $c$. In particular, there will be one such curve passing through every point on the $\beta = 3$ contour. The condition of local orthogonality between the vector $\mathbf{u}*$ and the limit-state, makes it clear that $\mathbf{u}*$ must be a point of tangency between the contour and the appropriate limit-state. This point must give the largest value of the demand on the contour. Thus, to find the controlling design situation for this value of $p_f$ we only need to search the contour for the point with the largest value of $d(u_1, u_2)$. If the system is designed with a capacity equal to this demand, then the design will be deemed adequate for all other demands with the same value of $p_f$.

Using the Rosenblatt transformation in (3), the $\mathbf{u}*$ design point can be mapped directly into $X$-space. If the demand depends on more than two $x_i$ components, the contour logic extends directly; i.e., $\beta$ is now the radius of an $N$-dimensional sphere. Clearly, though, the convenient visual nature of environmental contours is best served when $N = 2$.

### 2.2 Dynamic inverse FORM

The general load combination problem, includes multiple time-varying components, $X_i(t)$, each typically with different frequency content. This is the problem we seek to address here.

It has become customary to define the “characteristic” (or most probable) response as the level with an exceedance probability of $p_f = 1 - e^{-1}$. This is the Poisson estimate of the probability of failure if one failure is expected during the time interval $T$ in Eq. (2). From Eqs. (2) and (6) we find the characteristic value

$$ \beta_{cr} = [2 \ln(N)]^{1/2} $$

(10)

in which $N = v_0T$ is the average number of response cycles over the duration $T$ of interest.

We now consider a demand process $Z(t)$ that is a linear combination of two independent $U_i(t)$ processes:

$$ Z(t) = U_1(t) \cos(\theta) + U_2(t) \sin(\theta) $$

(11)

Because $U_1$ and $U_2$ are both standard Gaussian, any $(u_1, u_2)$ contour that could contain the characteristic extreme must approach the limiting values of $\beta_1 = [2 \ln(N_1)]^{1/2}$ and $\beta_2 = [2 \ln(N_2)]^{1/2}$ when $u_2 = 0$ and $u_1 = 0$, respectively. Here $N_1$ and $N_2$ are the average number of cycles of each response over the duration $T$ of interest. What remains in question, of course, is how the contour should be defined in intermediate cases. Leira (2012) has suggested an ellipsoidal contour:

$$ \left( \frac{u_1}{\beta_1} \right)^2 + \left( \frac{u_2}{\beta_2} \right)^2 = 1; \quad \beta_i = [2 \ln(N_i)]^{1/2} $$

(12)

for $i = 1, 2$ and the following sections will provide an alternate approach.

The formalism of FORM seeks results that are exact within the limitations of a
linearized limit state in U-space and the Poisson approximation of the crossing rate. Thus, the starting point is again the demand quantity \( Z(t) \) in Eq. (11). We consider \( U_1(t) \) and \( U_2(t) \) to be standard Gaussian processes, so \( Z(t) \) is also a standard Gaussian process. This implies that we can again use the standard result from Eq. (10) for its characteristic value:

\[
\zeta_{cr} = [2 \ln(N_Z)]^{1/2} \tag{13}
\]

in which \( N_Z \) now represents the average number of cycles of the combined response \( Z(t) \) during the time interval \( T \).

It only remains to estimate \( N_Z \). For this purpose, it is convenient to note that for a standard Gaussian process such as \( Z(t) \), the zero-crossing rate \( v_{Z0} \) is proportional to the rms \( \sigma_Z \) of the derivative process:

\[
\sigma_Z = \omega_{Z0} \sigma_Z = 2\pi v_{Z0} \sigma_Z = 2\pi v_0 \tag{14}
\]

Here \( v_{Z0} = \omega_{Z0} / (2\pi) \) is the mean frequency in Hz.

To use Eq. (14), we differentiate (11):

\[
\dot{Z}(t) = \dot{U}_1(t) \cos(\theta) + \dot{U}_2(t) \sin(\theta) \tag{15}
\]

The variances then satisfy

\[
\sigma_Z^2 = \sigma_{U_1}^2 \cos^2(\theta) + \sigma_{U_2}^2 \sin^2(\theta) \tag{16}
\]

and Eqs. (14) and (16) give

\[
N_Z = \left[ N_1^2 \cos^2(\theta) + N_2^2 \sin^2(\theta) \right]^{1/2} \tag{17}
\]

in which \( N_i \) is the average number of cycles of \( U_i(t) \) during the time interval \( T \).

We can use Eqs. (17), (13) and (8) with \( \beta(\theta) = \zeta_{cr} \) as a recipe to generate a \((u_1, u_2)\) contour, similar to the parametric form in Eq. Eqs.(10) and (11). This procedure is readily implemented by sweeping the angle \( \theta \) around a circle and plotting the (angle-dependent) radius \( \beta(\theta) \). It is clear that the proper limiting values of \( [2 \ln(N_1)]^{1/2} \) and \( [2 \ln(N_2)]^{1/2} \) values are found when \( \theta = 0 \) and \( \pi / 2 \), respectively.

There are at least two possible ways to use the dynamic FORM contour: 1) as stated earlier, the true FORM solution is to find the point \( \mathbf{u}^* \) on the contour such that the vector \( \mathbf{u}^* \) is orthogonal to the design limit passing through this point, and 2) a cruder, but conservative, approximation is to find the largest demand value anywhere on the contour. Note that these two approaches are the same for the static problem in which the contours are circular (or spherical), but they generally give different results in dynamic problems. This will be investigated in the numerical examples.

3 Numerical Example

For a ship example, Leira (2012) considered a combined vertical midspan bending moment response, \( Y(t) \), of the form

\[
Y(t) = X_1(t) + X_2(t) \tag{18}
\]

in which \( X_1(t) \) is the still-water bending moment, and \( X_2(t) \) is the wave-induced bending moment. It is further assumed that \( X_1 \) and \( X_2 \) have average cycle numbers of \( N_1 = 120 \) and \( N_2 = 10,000 \) over the duration of interest. Since each \( X_i(t) \) is a translation process, \( N_1 \) and \( N_2 \) also give the average number of cycles of the underlying Gaussian processes \( U_1(t) \) and \( U_2(t) \). Note that the \( N_i \) cycle count must be the rate of upcrossings of the median of \( X_i(t) \).

Figure 3 compares the \((u_1, u_2)\) contours generated in this case according to: 1) the ellipse of Leira in Eq. (12), and 2) the FORM-based contours of Eqs. (13)-(17).
in the limiting cases when either $U_1$ or $U_2$ goes to zero, Eq. (10) gives the characteristic extreme values as

$$
\beta_1 = \left[2\ln(120)\right]^{1/2} = 3.09 \\
\beta_2 = \left[2\ln(10000)\right]^{1/2} = 4.29.
$$

Both the contours in Fig. 3 show the appropriate values of $\beta_1 = [2\ln(N_1)]^{1/2} = 3.09$ for $\theta = 0$ and $\beta_2 = [2\ln(N_2)]^{1/2} = 4.29$ for $\theta = \pi/2$. For intermediate angles, however, the FORM-based contour consistently yields larger values of the radius $\beta$. The differences are especially pronounced when the "slow" process $U_1$ dominates. When a small contribution of the "fast" process $U_2$ is introduced (i.e., $\theta$ slightly greater than 0) the cycle number $N_2$ of the combined demand rises quickly, an effect that is not included in the ellipsoidal contour.

The plot also highlights the differences mentioned earlier between the two methods of using the FORM-based contour. The true (FORM-predicted) demand may or may not be the maximum demand anywhere on the contour. For greatest accuracy, the FORM-based contour should not be searched over its entirety, but rather evaluated at a specific angle $\theta$ relevant to the problem at hand. In some practical cases, however, the maximum demand on the FORM contour is a close approximation of the true FORM design point. This is shown in the examples that follow.

### 3.1 Case 1

Note that the $U$-space contours in Fig. 3 depend only on the number of cycles, $N_1$ and $N_2$ over the duration of interest. We will retain the values of $N_1 = 120$ and $N_2 = 10,000$ through all the examples in this section; thus, the contours in $U$-space will remain constant throughout.

To find the $h_1(u_1)$ functions that will transfer these contour to the physical $X$-space we must know the probability distributions of $X_1(t)$ and $X_2(t)$, which we will consider to be independent processes.

We will begin with one of the situations considered by Leira (2012). In particular we will use the following mean and standard deviation values:

- Moments of $X_1$: $m_1 = 1,000$ $\sigma_1 = 500$
- Moments of $X_2$: $m_2 = 500$ $\sigma_2 = 500$ (19)

with $X_1$ being Gaussian and $X_2$ exponentially distributed.

The Rosenblatt transformation between $x$ and $u$ is given by

$$
\begin{align*}
U_1 &= f_1(x_1) = (x_1 - m_1)/\sigma_1 \\
X_1 &= h_1(u_1) = m_1 + \sigma_1 u_1 \\
U_2 &= f_2(x_2) = \Phi^{-1}(1 - e^{-x_2/\sigma_2}) \\
X_2 &= h_2(u_2) = -\sigma_2 \ln[1 - \Phi(u_2)]
\end{align*}
$$

Figures 8 and 9 show the resulting contours (and limit states) in $U$-space and $X$-space, respectively.

We know that the critical point $u^* = (u_1^*, u_2^*)$ in $U$-space must correspond to a vector that is orthogonal to the contour curve at that point. One can estimate this point geometrically directly from the contour plot, perhaps concluding that the slope $u_2^*/u_1^*$
of the line to \( \mathbf{u}^* \) is approximately 4.0. One can obtain a more accurate estimation by using Eqs. (27) and (20) to calculate the slope of the contour at any point. In particular, we can note that an incremental movement in X-space with slope \( \delta x_2 / \delta x_1 \) corresponds to a movement in U-space with slope

\[
\frac{\delta u_2}{\delta u_1} = \frac{f_2'(x_2) \delta x_2}{f_1'(x_1) \delta x_1} = \frac{h_1'(u_1) \delta x_2}{h_2'(u_2) \delta x_1} \tag{21}
\]

A constant-y line in X-space has \( \delta x_2 = -\delta x_1 \) so that the slope of the U-space contour at point \( \mathbf{u}^* \) must be \(-h_1'(u_1^*)/h_2'(u_2^*)\). Thus, the orthogonal vector \( \mathbf{u}^* \) must have a slope of \( h_2'(u_2^*)/h_1'(u_1^*) \), and the numerical condition for the critical point can be written as

\[
\frac{u_2^*}{u_1^*} = \frac{h_2'(u_2^*)}{h_1'(u_1^*)} \tag{22}
\]

It is found that the point on the contour satisfying this relationship has \( \theta^* = 77.4^\circ \) and \( \beta^* = 4.29^\circ \), giving \( \mathbf{u}^* = (0.935, 4.48) \). The mapping into X-space gives \( \mathbf{x}^* = (1470, 5570) \) so that \( y_{cr} = 7040 \).

This situation—where the “fast” process dominates—is one where the Leira and FORM contours show the greatest agreement. Thus, the FORM contour value of \( y_{cr} = 7040 \) is just slightly above the value of \( y_{cr} = 6950 \) from the ellipsoidal contour. It is also found that the maximum y value on the FORM contour almost coincides with the FORM design point.

### 3.2 Dominant still-water moment

We now modify the statistics in Eq. (19) so that the still-water bending moment, \( X_1(t) \), has greater effect. Specifically, we rescale \( X_1(t) \) by a factor of 3, and \( X_2(t) \) by a factor of 0.5, which is also a situation studied by Leira. We now have

\[
\begin{align*}
\text{Moments of } X_1 & : m_1 = 3000 \quad \sigma_1 = 1500 \\
\text{Moments of } X_2 & : m_2 = 250 \quad \sigma_2 = 250 \tag{23}
\end{align*}
\]

in the Rosenblatt transformation of Eq. (20).

The contours for this situation are shown in Figs. 10 and 11. The orthogonality condition in (22) gives \( \theta^* = 13^\circ \), \( \beta^* = 3.93^\circ \), and \( \mathbf{u}^* = (3.83, 0.88) \). The mapping into X-space gives \( \mathbf{x}^* = (8740, 417) \) so that...
This can be compared with a maximum value of $y$ of 9350 on the FORM contour and 8100 on the elliptical contour. Thus, the maximum value on the FORM contour is close to the true FORM solution, while the elliptical contour gives a design value that is significantly lower, suggesting that the result may be nonconservative.

### 3.3 Gaussian $X$

Next we consider situations with the same parameter values as were used in Secs. 3.1 and 3.2, but now using a Gaussian distribution for $X_2(t)$ as well as $X_1(t)$.

It may be noted that there is no real need for a method such as FORM for this totally Gaussian problem. In particular, $Y = X_1 + X_2$ is also Gaussian, and its moments are

$$m_Y = m_1 + m_2 \quad \sigma_Y = \left(\sigma_1^2 + \sigma_2^2\right)^{1/2}$$

In a procedure analogous to Eq. (10), but now shifted and rescaled to reflect the mean $m_Y$ and standard deviation $\sigma_Y$ of $Y$, we can say that the characteristic value of $y$ is

$$y_{cr} = m_Y + \sigma_Y \left[2 \ln(N_Y)\right]^{1/2}$$

and $N_Y$ is given, as in Eqs. (14)-(17), as

$$N_Y = \left(\frac{N_1^2 \sigma_1^4 + N_2^2 \sigma_2^4}{\sigma_1^4 + \sigma_2^4}\right)^{1/2}$$

Thus, the design point, or characteristic value, of $Y$ can be evaluated directly from Eqs. (25) and Eq. (26).

One reason for including this totally Gaussian problem in this investigation is to confirm the FORM method described above. The FORM-based contour results should be in complete agreement with the direct solution, since the FORM approximation involves a linear combination of Gaussian demand terms. Comparisons of the results from an exact solution and those obtained by using an elliptical contour or the maximum $Y$ value on the FORM contour are also useful in determining the usefulness of these methods.

The Rosenblatt transformation for both $X_1$ and $X_2$ is now given by

$$u_i = f(x_i) = (x_i - m_i)/\sigma_i$$

$$x_i = h_i(u_i) = m_j + \sigma_j u_j$$

The limit state in $U$-space can be written as

$$u_2 = f(y-x_1) = f[y-h_1(u_1)]$$

$$= (y-m_1-m_2-\sigma_1 u_1)/\sigma_2$$
This straight line obtained directly from the Rosenblatt transformation, is also the FORM limit state. The orthogonality condition in Eq. (22) gives $u_2^*/u_1^* = \sigma_2/\sigma_1$. This clearly makes the vector $u^*$ orthogonal to the limit state with slope $-\sigma_1/\sigma_2$.

Using $\theta^* = \tan^{-1}(\sigma_2/\sigma_1)$ in Eq. (17) gives $N_Z$ as identical to the $N_Y$ in Eq. (26). Using Eq. (8) with this value of $\theta^*$ and Eq. (28) to convert then gives the X-space design point as the same as $y_{cr}$ from Eqs. (25) and (26). This confirms the anticipated agreement of FORM-based contour results and the direct solution for the totally Gaussian problem.

For the parameter values of Sec. 3.1 the limit state has a slope of $-1$, and the vector $u^*$ has a slope of $+1$ ($\theta^* = 45^\circ$). The design point from Eqs. (25) and (26) or from FORM is found to be $y_{cr} = 4480$. The results in U-space and X-space are shown in Figs. 4 and 5,
respectively. In contrast, the elliptical contour is shown to be non-conservative, yielding the estimate of \( y_{cr} = 4150 \). To put this difference into better perspective, it is convenient to focus on the dynamic response by removing the deterministic portion of the response (i.e., the \( m_Y \) term). This gives values of \( y_{cr} - m_Y \) of 2980 and 2650 for the exact result and the elliptical approximation, respectively. This is a response underestimation of 11%. The maximum value of \( y \) on the FORM contour is found to be in almost identical to the exact result for this situation.

Proceeding as above, the parameter values from Sec. 3.2 give the limit-state slope as \(-6\), and \( \theta = 9.46^\circ \). The design point from Eqs. (25) and (26) or from FORM is found to be \( y_{cr} = 9100 \). The results in U-space and X-space are shown in Figs. 6 and 7, respectively.

As before, the elliptical contour is non-conservative, yielding \( y_{cr} = 8050 \). Excluding the mean value \( m_Y = 1750 \) from each of the \( y_{cr} \) values shows an underestimation of about 18% from using the elliptical contour. The maximum value over the FORM contour is about 3.4% higher than the exact solution in this case.

4 Non-FORM Alternatives

Two calculations that do not use FORM approximations will now be presented. The first of these differs from the FORM approach only by evaluating crossings of the true X-space limit state, rather than a linear approximation in U-space. This procedure generally involves numerical integration, and is quite inefficient compared to FORM. It is introduced here only to investigate the accuracy of the FORM approximation for a system of the type investigated in Sec. 3. The second non-FORM approach is to replace the translation-process assumption that \( \hat{U}(t) \) and \( \hat{U}(t) \) are independent with an assumption of independence of \( X(t) \) and \( \dot{X}(t) \), which is not true when \( X(t) \) is a translation process.

The problems investigated here, as in Sec. 3, have a demand function of \( Y(t) = X_1(t) + X_2(t) \), but the number of crossings is across the true line \( Y = y \) in X-space. As before, \( X_1(t) \) and \( X_2(t) \) are considered to be independent processes. The exact rate of crossings (based on the Poisson approximation) of any level \( y \) can be written directly as a triple integral over the variables \((\hat{x}_1,\hat{x}_2,x_1)\). One way to simplify this rather cumbersome result is to use the Rosenblatt mapping into U-space and the resulting condition that \( U_1(t) \) and \( U_2(t) \) are independent Gaussian processes. This allows simplification to a single integral of

\[
N_Y(y) = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{(u_1^2 + u_2^2)/2}{2}} h_1'(u_2) \times \left( N_1^2[h_1'(u_1)]^2 + N_2^2[h_2'(u_2)]^2 \right)^{1/2} du_1
\]

in which \( u_2 = f_2[y-h_1(u_1)] \) and the \( f_i(x_i) \) and \( h_i(u_i) \) functions come from the Rosenblatt mapping. Numerical integration is generally required for evaluation of this final integral.

There are important special cases in which we may be interested in non-Gaussian processes that are not translation processes. For example, there is a non-trivial class of dynamic models for which the response gives the random variable \( X(t) \) as being Gaussian and independent of the random variable \( Y(t) \) at the same instant of time. We will give some limited results for this class of problems.

Again, we limit our attention here to the special case in which the demand is \( Y(t) = X_1(t) + X_2(t) \). In addition, we say that \([\dot{X}_1(t),\dot{X}_2(t)]\) is a pair of Gaussian random variables, and \([X_1(t),\dot{X}_1(t),X_2(t),\dot{X}_2(t)]\) is a set of independent random variables for any given value of \( t \). The variance expression for

\[
V(Y) = \int \left( N_1^2[h_1'(u_1)]^2 + N_2^2[h_2'(u_2)]^2 \right)^{1/2} du_1
\]
In particular, was considered for the ship the range of parameters is much broader than these calculations have been limited to the give questionable values of the design point. We have made additional numerical calculations in an attempt to identify situations in which any of the calculation techniques may give values. In Eq. (30) in the standard expression for the number of crossings by \( Y(t) \) gives

\[
N_Y(y) = (2\pi)^{1/2} \frac{p_Y(y)}{\sigma_Y} \left( \frac{N_1^2}{p_{X_1}(x_{1m})} + \frac{N_2^2}{p_{X_2}(x_{2m})} \right)^{1/2}
\]

As before, the design point is chosen to have \( N_Y(y) = 1 \), so that one can find this point by searching the expression in Eq. (31) over a sequence of \( y \) values. The only remaining difficulty is in finding the pdf of \( Y(t) \). This is generally quite simple for the situation considered here with only two \( X_i(t) \) components. In particular, one can always use numerical integration of \( p_{X_1}(x_1)p_{X_2}(y - x_1) \) if an analytical solution is not available.

5 Extended Numerical Comparisons

We have made additional numerical calculations in an attempt to identify situations in which any of the calculation techniques may give questionable values of the design point. These calculations have been limited to the situation of the demand being \( Y = X_1 + X_2 \) with Gaussian- \( X_1 \) and exponential- \( X_2 \), but the range of parameters is much broader than was considered for the ship-bending problem. In particular, \( \sigma_2/\sigma_1 \) was varied from \( 10^{-3} \) to \( 10^3 \) and \( N_2/N_1 \) varied from \( 10^{-4} \) to \( 10^4 \). The values of both \( N_1 \) and \( N_2 \) were restricted to be no smaller than 100. The focus was on situations in which \( N_j > N_j \) when \( \sigma_j < \sigma_j \), since these are situations in which the component with small variance can still have a significant effect on \( p_f \), since it contributes significantly to the crossing rate for \( Y \).

The first issue studied relates to the accuracy of the inverse FORM method. Over the very broad range of situations it was found that the FORM results were in very good agreement with the numerical integration results of Eq. (29). The discrepancy was usually less than 1%, although there were limited situations for which it was slightly larger. The largest discrepancy observed was about 4%, and occurred for the rather extreme case of \( \sigma_1 = \sigma_2, N_2 = 1000N_1 \). Over all, it seems that the inverse FORM method gives very accurate results for this problem.

The next numerical comparison was between the full FORM method and the simpler choice of seeking the largest demand anywhere on the \( p_f \)-contour. Recall that there was little difference between these methods for the ship-bending problem. It was found, though, that the difference could be much larger in situations where one of the components had a very small value of \( \sigma \) and a very large value of \( N \). For \( \sigma_2 = 10^{-3}\sigma_1, N_2 = 10^3N_1 \) the maximum-\( Y \) approach gave the design point about 38% larger than the FORM result. Similarly, \( \sigma_1 = 10^{-3}\sigma_2, N_1 = 10^3N_2 \) gave a 76% discrepancy. Of course, the maximum-\( Y \) approach is always conservative.

The final comparison is between the FORM (or numerical integration) result for the translation-process model and the result for the model in which \( X(t) \) is Gaussian and independent of \( X(t) \). It was found that the results from the two models often in very
Design Contours for Load Combinations
L.D. Lutes, S.R. Winterstein

good agreement, but the independent \( \dot{X} \) model sometimes gave smaller values of the design point. The largest difference observed was about 20%, and it occurred when \( \sigma_2 \geq 100\sigma_1 \), \( N_2 \leq 0.01\sigma_1 \). A similar, but smaller, difference of about 15% was observed for \( \sigma_2 = \sigma_1 \), \( N_2 > N_1 \). It should be noted that the differences in these calculations do not reflect an error in either model, but rather the effect of model choice. The translation model sometimes gives somewhat larger values of the design point than does the independent \( \dot{X} \) model. Thus the usual FORM approach with the translation model would be conservative in the situation in which \( \dot{X}(t) \) was actually independent of \( X(t) \).

6. Summary and Conclusions
The inverse FORM method, previously used for static problems, has been extended to apply to dynamic situations. The method is based on the use of probability-of-failure contours to identify design points. It has been shown that the FORM contour sometimes gives results that are significantly more conservative than those obtained by using an elliptic contour. The full FORM approach is not based on choosing the maximum-demand point on the FORM contour, although that is a simple alternative. Numerical results have shown that the new technique agrees very well with inefficient numerical integration. For a specific problem of bending moment in a ship the FORM results agree well with the simpler maximum-demand approach, but it has been shown that that maximum-demand is overly conservative in some more extreme situations.

A brief investigation has been made of a non-FORM model in which \( \dot{X}(t) \) is Gaussian and independent of \( X(t) \). For the system studied, numerical results have shown that this independent-derivative model sometimes gives significantly smaller demand values than inverse FORM, so that use of inverse FORM is conservative in situations in which the other model is more accurate.

References