EXTREMES OF NONLINEAR VIBRATION:
MODELS BASED ON MOMENTS, L-MOMENTS, AND MAXIMUM ENTROPY

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ABSTRACT
Nonlinear effects beset virtually all aspects of offshore structural loading and response. These nonlinearities cause non-Gaussian statistical effects, which are often most consequential in the extreme events—e.g., 100- to 10,000-year conditions—that govern structural reliability. Thus there is engineering interest in forming accurate non-Gaussian models of time-varying loads and responses, and calibrating them from the limited data at hand.

We compare here a variety of non-Gaussian models. We first survey moment-based models; in particular, the 4-moment “Hermite” model, a cubic transformation often used in wind and wave applications. We then derive an “L-Hermite” model, an alternative cubic transformation calibrated by the response “L-moments” rather than its ordinary statistical moments. These L-moments have recently found increasing use, in part because they show less sensitivity to distribution tails than ordinary moments.

Here we derive a new “L-Hermite” model of random vibration—an alternative cubic transformation calibrated by the response “L-moments” rather than its ordinary statistical moments. We find here, however, that these L-moments may not convey sufficient information to accurately estimate extreme response statistics. Finally, we show that 4-moment maximum entropy models, also applied in the literature, may be inappropriate to model broader-than-Gaussian cases (e.g., responses to wind and wave loads).

INTRODUCTION
Nonlinear effects beset virtually all aspects of offshore structural loading and response. These nonlinearities cause non-Gaussian statistical effects, which are often most consequential in the extreme events—e.g., 100- to 10,000-year conditions—that govern structural reliability. Thus there is engineering interest in forming accurate non-Gaussian models of time-varying loads and responses, and calibrating them from the limited data at hand.

We compare here a variety of non-Gaussian models. We first survey moment-based models; in particular, the 4-moment “Hermite” model. This models the non-Gaussian response \( x(t) \) as a cubic transformation, either to or from a Gaussian process \( u(t) \), as a cubic polynomial [1]. This model, and variants that use other “parent” variables than Gaussian, have commonly been used in wind and wave applications [2].

We then consider models based on L-moments [3]. These L-moments have recently found increasing use, in part because they show less sensitivity to distribution tails than ordinary moments. In offshore engineering, L-moment models have been applied to consider wave runup [4] and Morison drag loads [5].

Here we derive a new “L-Hermite” model of random vibration—an alternative cubic transformation calibrated by the response “L-moments” rather than its ordinary statistical moments. We find, however, that these L-moments may not convey sufficient information to accurately estimate extreme response statistics. Finally, we show that 4-moment maximum entropy models, also applied in the literature (e.g., [6], [7]), may be inappropriate to model broader-than-Gaussian cases (e.g., responses to wind and wave loads).

Scope and Organization
As noted above, we will explore three different non-Gaussian models: a Hermite model based on conventional moments, a (new) Hermite model based on L-moments, and finally a 4-moment maximum entropy model. Through a set of examples, we study how well these different models represent the distribution tails of interest. Because L-moments are relatively new, we
begin with a brief description of L-moments and how they differ from ordinary moments.

**GENERAL RESULTS FOR L-MOMENTS**

This section closely follows the work of Hosking [3], where much additional information can be found. Perhaps the simplest way to view L-moments is in terms of an ordered sample of size n \((X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n})\), drawn from the distribution of X. The n-th L-moment, \(\lambda_n\), is then defined as a linear combination of the order statistics \(E[X_{i:n}]\). In particular, the first four L-moments are

\[
\begin{align*}
\lambda_1 &= E[X] \\
\lambda_2 &= \frac{1}{2} E[X_{2:2} - X_{1:2}] \\
\lambda_3 &= \frac{1}{3} E[X_{3:3} - 2X_{2:3} + X_{1:3}] \\
\lambda_4 &= \frac{1}{4} E[X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}]
\end{align*}
\]

Clearly, \(\lambda_1\) and \(\lambda_2\) are measures of central trend and dispersion. Higher L-moments reflect different aspects of distribution shape. In terms of the CDF of X, \(F(x)\), \(E[X_{i:n}]\) is the estimated CDF value associated with \(x_i\); e.g., \(\hat{F}_i=i/N\). One may instead use other “plotting point” locations, of the general form \(\hat{F}_i=(i+\gamma)/(N+\delta)\) for \(\delta > \gamma > -1\).

\[
\lambda_n = \frac{1}{N} \sum_{i=1}^{N} x_i w_n(\hat{F}_i) 
\]

Here \(\hat{F}_i\) is the estimated CDF value associated with \(x_i\); e.g., \(\hat{F}_i=i/N\). One may instead use other “plotting point” locations, of the general form \(\hat{F}_i=(i+\gamma)/(N+\delta)\) for \(\delta > \gamma > -1\).

**L-Moments for Gaussian Variables**

Consider now the special case of a standard normal variable, commonly denoted \(U\), with cumulative distribution function \(F(u) = \Phi(u)\) and probability density function \(\phi(u) = \exp(-u^2/2)/\sqrt{2\pi}\). From Eqn. 5, its L-moments are of the form

\[
\lambda_n = \int_{-\infty}^{\infty} u \cdot w_n[\Phi(u)]\phi(u)du = E[Uw_n(\Phi(U))] 
\]

These weight functions are given explicitly in Eqns. 32–34. Because \(w_1(u)\) and \(w_3(u)\) are even functions of \(u\), \(uw_1(u)\) and \(uw_3(u)\) are odd so that \(\lambda_1=\lambda_3=0\) in Eqn. 11. The non-zero L-moments, \(\lambda_2\) and \(\lambda_4\), are evaluated to be

\[
\lambda_2[U] = \frac{1}{\sqrt{\pi}} = 0.56419; \quad \lambda_4[U] = 0.06917 
\]

The corresponding L-skewness and L-kurtosis, \(\tau_3=\lambda_3/\lambda_2\) and \(\tau_4=\lambda_4/\lambda_2\), are then

\[
\tau_3 = 0; \quad \tau_4 = \frac{0.06917}{0.56419} = 0.1226 
\]

From Eqn. 11, note that \(\lambda_n=E[Uw_n(\Phi(U))]\), the expected product of \(U\) and the weight function \(w_n(\Phi(U))\). Figure 1 shows the behavior of this product, \(L_n(\Phi(U))\), for \(n=3\) and 4. It is clear that \(L_n(u)\), and hence \(\lambda_n\), gives much less weight to tail values than \(u^3\) and \(u^4\), the weighting functions for standard moments of orders 3 and 4. In particular, in the tails the weight functions in Eqns. 6–9 approach 1 in absolute value, so that extreme outcomes are weighed roughly linearly by the L-moments, rather than to the third and fourth powers by skewness and kurtosis. (This is sensible in that L-moments are linear combinations of order statistics—hence their name—and, unlike \(\mu_n=E[(X-m_X)^n]\), all L-moments retain the units of \(X\).) This tail-insensitivity of L-moments will be shown below to be a drawback, when one fits models to these moments to estimate extremes.
TRANSFORMATION MODELS 1: HERMITE MODELS

Hermite models are transformations of the form \( X = g(U) \), in which \( g \) is a cubic function rearranged in terms of the Hermite polynomials \( \text{He}_2(U) = U^2 - 1 \) and \( \text{He}_3(U) = U^3 - 3U \):

\[
X = m_X + \sigma_X \kappa [U + c_3(U^2 - 1) + c_4(U^3 - 3U)]
\]

in which \( U \) is standard normal, and \( m_X \) and \( \sigma_X^2 \) are the mean and variance of \( X \). We consider here only “softening” cases, whose kurtosis \( \alpha_4 \) exceeds 3, the value in the Gaussian case. (For “harden-\( g \)ing” cases in which \( \alpha_4 < 3 \), the roles of \( X \) and \( U \) are inter-changed, using the cubic transformation to expand the tails of \( X \) to achieve Gaussianity. We believe this use of dual models greatly enhances modelling flexibility.)

By using Hermite polynomials in Eqn. 14, the quantity in square brackets has zero mean and uncorrelated terms. Its variance is then \( 1 + c_3^2E[\text{He}_2(U)^2] + c_4^2E[\text{He}_3(U)^2] \), or simply \( 1 + 2c_3^2 + 6c_4^2 \). Thus, ensuring Eqn. 14 to have consistent variance requires

\[
\alpha = \frac{1}{\sqrt{1 + 2c_3^2 + 6c_4^2}}
\]

It remains to select the constants \( c_3 \) and \( c_4 \) to be consistent with the skewness \( \alpha_3 \) and kurtosis \( \alpha_4 \) of \( X \). This is the topic of the remainder of this section.

We first consider the case in which \( c_3 = 0 \), so that \( X \) is symmetrically distributed about its mean. The kurtosis of \( X \) in this case is

\[
\alpha_4 = \frac{E[(X - m_X)^4]}{\sigma_X^4} = \frac{3 + 24c_4 + 252c_4^2 + 1296c_4^3 + 3348c_4^4}{(1 + 6c_4^2)^2}
\]

Equation 16 gives an implicit result for \( c_4 \), the cubic coefficient required to match the kurtosis \( \alpha_4 \) found from a given model or dataset. For small deviations from Gaussianity, \( c_4 \) will be small and explicit approximations for \( c_4 \) are possible. The simplest, “first-order” result retains only linear terms in \( c_4 \) from Eqn. 16:

\[
\alpha_4 = 3 + 24c_4; \quad c_4 = \frac{\alpha_4 - 3}{24}
\]

The more standard, “second-order” Hermite model more accurately captures kurtosis, by also retaining quadratic terms in \( c_4 \) (in both numerator and denominator of Eqn. 16):

\[
\alpha_4 = 3 + 24c_4 + 216c_4^2; \quad c_4 = \frac{-1 + \sqrt{1 + 1.5(\alpha_4 - 3)}}{18}
\]

Equations 18–19, together with Eqns. 14–15, form the basis of the standard, “second-order” Hermite model.

Most recently, we use numerical routines to obtain “exact” values of \( c_3 \) and \( c_4 \) from constrained optimization, minimizing errors in matching moments under the constraint that the Hermite transformation remains monotonic. (Newton-Raphson tech-\( niques have also been suggested [8] to estimate these coefficients.) Commonly these routines reproduce the specified moments to the tolerance requested. These are the source of the Hermite results shown here. Note too that analytical fits have also been made [2] to these “exact” \( c_3 \), \( c_4 \) values:

\[
c_3 = \frac{\alpha_3 - 0.015(\alpha_4 - 3) + 0.3\alpha_4^2}{6}
\]

\[
c_4 = \frac{\alpha_4[1 - 1.43\alpha_2^2(\alpha_4 - 3)^{-1} - 0.1\alpha_4^2]}{10}
\]

To test the accuracy of the approximations in Eqns. 17–22, Figure 2 compares their kurtosis predictions with the exact result in the symmetric case (Eqn. 16). The “third-order” results in this figure correspond to Eqns. 20–22.
TRANSFORMATION MODELS 2: L-HERMITE MODELS

We now seek to derive new models, again adopting a cubic Hermite transformation (Eqn. 14) now calibrated by L-moments. To calculate L-moments of Eqn. 14, it is first useful to rearrange terms. Regrouping $U + c_4(U^3 - 3U)$ as $(1 - 3c_4)U + c_4U^3$ and dividing by $(1 - 3c_4)$, one finds the equivalent representation

$$X = m_X + K[U + b(U^2 - 1) + cU^3]$$ (23)

The benefit here is that the highest-order term is simplified to standard normal process, $\lambda$, while the Hermite model decouples: the desired skewness and kurtosis. In contrast, the L-moments of an analytical Hermite model that may only approximately match proximate results for these coefficients (Eqns. 18–19), and hence $\sigma_X^2$, the scaling factor $K$ now becomes

$$K = \frac{\sigma_X}{\sqrt{1 + 2b^2 + 6c + 15c^2}}$$ (24)

In terms of the original coefficients $c_3$ and $c_4$, the new coefficients are $b = c_3/(1 - 3c_4)$ and $c = c_4/(1 - 3c_4)$.

Our goal now is to calibrate Eqn. 23; i.e., choose $b$ and $c$ that yield a specified set of $(\tau_3, \tau_4)$ values. Appendix 1 shows that this leads to the results

$$b = \frac{9.21\tau_3}{11.68 - 2.5\gamma}; \quad c = \frac{\gamma - 1}{11.68 - 2.5\gamma}; \quad \gamma = \frac{\tau_4}{\tau_{4,\text{gauss}}}$$ (25)

in which $\tau_{4,\text{gauss}}=0.1226$ (Eqn. 13).

Equations 23–25 comprise the L-moment version of the Hermite model—referred to below as the “L-Hermite” model. Note that to be consistent with the Hermite model we have chosen the scaling factor $K$ in Eqn. 24 to preserve $\sigma_X^2$; if we instead wish to preserve $\lambda_2$, we find the alternative choice

$$K = \frac{\sqrt{\pi}\lambda_2}{(1 + 2.5c)} = \frac{1.77\lambda_2}{(1 + 2.5c)}$$ (26)

The simplicity of these results is notable. The central moments, $\mu_m = E[(X - m)^m]$, of Eqn. 14 yield coupled results: both $\mu_3$ and $\mu_4$ vary with both coefficients, $c_3$ and $c_4$. This leads to approximate results for these coefficients (Eqns. 18–19), and hence an analytical Hermite model that may only approximately match the desired skewness and kurtosis. In contrast, the L-moments of the Hermite model decouple: $\lambda_3$ depends only on $b$ in Eqn. 23, while $\lambda_4$ depends only on $c$. The results (Eqn. 25) permit the L-Hermite model to preserve the L-moment ratios, $\tau_3$ and $\tau_4$, without approximation.

EXAMPLE 1: SYMMETRIC TRANSFORMATIONS

We first consider $X(t)$ as a symmetric transformation of a standard normal process, $U(t)$:

$$X(t) = g(U(t)) = U(t) + c|U(t)|^{m-1}U(t); \quad m = 2, 3, 4, ...$$ (27)

For fluid loads, the quadratic case ($m=2$) corresponds to the standard Morison drag load model. Higher $m$ values reflect higher-order models.

In general, the index $m$ controls the tail behavior of $X$: $|X|$ grows like $|U|^m$ for large $|U|$. The coefficient $c$ determines the relative importance of this nonlinear term; i.e., where in the distribution tails this term begins to dominate. For given $m$, the shape parameter $c$ can be related to either the kurtosis, $\alpha_4$, or the L-kurtosis, $\tau_4$. Such results are discussed in Appendix 1, and summarized in Table 2 for $2 \leq m \leq 5$.

Our main goal here is to represent any asymmetric nonlinear system by either its kurtosis, $\alpha_4$, or its L-kurtosis, $\tau_4$. It is thus useful to compare different models, calibrated to have the same fourth moment or L-moment, to see what variability remains. We hope this remaining variability to be small; that is, that the fourth moment goes a long way toward “explaining” the tail behavior of a nonlinear system, regardless of the precise form of its nonlinearity.

Figures 3–4 show that for kurtosis-based models, this is generally the case. These show the mean upcrossing rate of $X(t)$, $\nu_X(x)$, for the various transformed Gaussian models in Table 2. In general, for any transformed Gaussian process $X(t) = g(U(t))$ we find

$$\nu(x) = \nu_0 \exp(-u^2(x)/2); \quad u(x) = g^{-1}(x) = \Phi^{-1}[F(x)]$$ (28)

where $\nu_0$ is the upcrossing rate of the median of $X(t)$, and $F$ is the CDF of $X(t)$. All models in these figures have been calibrated—that is, their $c$ values chosen—to have a specific kurtosis value:
As may be expected, these models eventually diverge, and models of higher order (larger $m$) have PDFs with broader tails, and hence higher rates of upcrossings. Nonetheless, by preserving the fourth moment, the models cluster notably, yielding similar results to rates of about $\nu(x)/\nu(0)=10^{-3}$. This is particularly significant because there are on the order of 1000 cycles in a typical stationary, 3-hour seastate (number of 10-second waves in 3 hours=1080). Thus, four-moment models appear here to describe the tails sufficiently for practical purposes of extreme value analysis of marine structures.

In contrast, models fit here to L-moments do not define the response tails with comparable accuracy. Figures 5–6 show similar upcrossing rates, now found by preserving the fourth L-moment, $\tau_4$. Specifically, these results use the values $\tau_4=0.185$ and .220, which are roughly consistent with the cubic model when $\alpha_4=5$ and 7, respectively. Thus, the results for the cubic
model in Figs. 5–6 are similar to those in Figs. 3–4.

Most notably, different models with the same $\tau_4$ yield markedly different tail behavior, exhibited here at crossing rates of about $V(x)/\nu_0=10^{-2}$. Thus, the benefit of L-moments—their tail-insensitivity—is also their weakness: model uncertainty here begins to arise an order of magnitude more frequently—at levels crossed every 100 cycles rather than 1000—compared to 4-moment Hermite models.

EXAMPLE 2: LOGNORMAL MODELS

To test asymmetric cases, we consider the lognormal process $X(t)$, for which

$$X(t) = g(U(t)) = x_{50} \exp(\sigma_{lnX} U(t)) ; \quad \sigma^2_{lnX} = \ln(1 + V^2_X)$$

(29)

in which $x_{50}$ and $V_X$ are the median and COV (coefficient of variation) of $X(t)$. Figures 7–8 show results for $V_X=0.5$ and $V_X=1.0$, for which $(\alpha_3, \alpha_4)$ are (1.63, 8.04) and (4.00, 41.0) respectively. Findings here are similar to those in Example 1. Even for the extremely non-Gaussian case when $V_X=1.0$, a 4-moment fit shows good accuracy to about $V(x)/\nu_0=10^{-3}$. In contrast, fits to 4 L-moments again begin to diverge from exact results at around $V(x)/\nu_0=10^{-2}$. (The 4-moment fits here use “exact Hermite” models; i.e., Eqn. 14 with $c_3, c_4$ chosen to give exact $\alpha_3, \alpha_4$ values.)

MAXIMUM ENTROPY MODELS

Finally, we consider another model suggested for non-Gaussian processes: the “maximum entropy” model [6]. The resulting probability density of $X(t)$, assuming four moments are known, is of the form

$$f(x) = \exp(-\kappa(x)) ; \quad \kappa(x) = \sum_{n=0}^{4} k_n x^n$$

(30)

The coefficients $k_1, \ldots, k_4$ are chosen to preserve (or minimize error in) the four moments. Unit area is achieved through $k_0$.

Most critically, the large-$\alpha$ behavior of Eqn. 30 is asymptotically given by its highest-order term. Thus, $f(x)$ will ultimately decay like $\exp(-k_4 x^4)$ as $|x| \to \infty$. This implies that

1. $k_4 \geq 0$ so that $f(x)$ converges as $|x| \to \infty$, and
2. because $k_4 \geq 0$, $f(x)$ will ultimately decay at least as fast as the Gaussian density.

This makes the model of questionable use for “softening cases” ($\alpha_4 > 3$), the most common practical case of interest.

**Example 1 Revisited.** We first revisit example 1, for which we require that $f_X(x)$ be symmetric—hence $k_1=k_3=0$. Because $k_4 \geq 0$, Eqn. 30 must lead here to a “hardening” non-Gaussian model (with kurtosis $\alpha_4 \leq 3$). In fact, in this case Eqn. 30 coincides with the exact result for a “Duffing oscillator,” which includes a cubic hardening spring. Because our example 1 cases require $\alpha_4 > 3$, there is no maximum entropy solution in these cases. (Of course, a “softening” model with $k_4 < 0$ can be forced if Eqn. 30 is truncated at a finite upper-bound $x_{max}$. However, all results will then depend upon the user-defined value of $x_{max}$, required to reconcile the inappropriate functional form—hardening in Eqn. 30—with the actual softening behavior.)
Example 2 Revisited. We now revisit the lognormal cases in Example 2. In contrast to Example 1, the positive skewness values here yield negative $k_3$ in Eqn. 30, which expands the right tail of $f_X(x)$ from the Gaussian model and hence can also give $\alpha_4 > 3$. However, as noted above we still require positive $k_4$, so that these cases (and many others) yield $(k_3,k_4)$ values of opposing signs. These opposing effects—and the resulting bimodal PDFS—are clearly shown in Figs. 9–10. PDF results begin to diverge from exact values when $f_X(x)/\max[f_X(x)]$ has fallen off to about $10^{-2}$. Because $f_X(x)$ and $v(x)$ are roughly proportional—the proportionality is exact if $X$ and $\dot{X}$ are independent—this suggests that maximum entropy fails at a level similar to that of L-moment models. These failures, of course, have completely different causes: maximum entropy fails due to an inappropriate functional form, while L-moment models fail because their parameters are insufficiently tail-sensitive.

Example 3. Because of its wide study in the literature (e.g., [9], [10]), we consider a final case in which wind loads are applied to a 1DOF structure. The structural motion $X(t)$ satisfies

$$\ddot{X} + 2\zeta\omega_n\dot{X} + \omega_n^2 X = Y(t)^2$$

in which $Y(t)$ is a normalized wind velocity process, assumed here to be a Gaussian process. Following the cited references, we assume here that $\omega_n=1.26$ [rad/sec], $\zeta=.30$ (including viscous drag), and the covariance between $Y(t)$ and $Y(t+\tau)$ is $\exp(-0.12|\tau|)$. The response moments are then $\alpha_3=2.7$ and $\alpha_4=14.3$, suggesting notable non-Gaussian behavior.

Figure 11 shows the distribution of $X$, estimated by simulation, on normal probability scale. Also shown is a two-moment Gaussian fit, which, as may be expected, dramatically underestimates upper response fractiles of practical interest. The cubic Gaussian model (Hermite model with exact 4 moments) is a marked improvement, showing good agreement far into the response tails. In contrast, the maximum entropy model is found inconsistent, due to its ultimate hardening nature noted above. It thus underestimates response fractiles $x_p$ systematically for $p$ above .999 (exceedance probabilities below $10^{-3}$).

SUMMARY

A range of non-Gaussian models have been surveyed. We have first reviewed the 4-moment Hermite model. The Hermite
Table 1. Estimated 3-hour extreme, \(x_{3\ hr}\), from different methods divided by exact value. All results assume \(V(x_{3\ hr})/N_0=10^{-3}\). Maximum entropy results also assume that \(V(x)\) and \(f(x)\) are proportional.

<table>
<thead>
<tr>
<th>Strength of Nonlinearity:</th>
<th>Moderate ((\alpha_4=5.0; \tau_4=.185))</th>
<th>Strong ((\alpha_4=7.0; \tau_4=.220))</th>
<th>Moderate ((\text{COV}=0.5))</th>
<th>Strong ((\text{COV}=1.0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hermite:</td>
<td>0.94–1.07</td>
<td>0.95–1.08</td>
<td>0.98</td>
<td>1.00</td>
</tr>
<tr>
<td>L-Hermite:</td>
<td>0.65–1.18</td>
<td>0.61–1.25</td>
<td>0.93</td>
<td>0.76</td>
</tr>
<tr>
<td>Max Entropy:</td>
<td>No Solution Available</td>
<td></td>
<td>1.01</td>
<td>0.79</td>
</tr>
</tbody>
</table>

The table summarizes the results of Figs. 3–10. It focuses on the maximum response, \(x_{3\ hr}\), in a 3-hour seastate. Assuming this seastate comprises \(10^5\) cycles, \(x_{3\ hr}\) is defined here as \(V(x_{3\ hr})/N_0=10^{-3}\). For example, Fig. 3 shows that for a polynomial model with \(\alpha_4=5.0\), exact values of \(x_{3\ hr}\) range from \(5.4\sigma_x \leq 6.2\sigma_x\), for \(2 \leq n \leq 5\). Because the Hermite model predicts \(5.8\sigma_x\), it leads to ratios of predicted/exact \(x_{3\ hr}\) ratios of \(5.8/(5.4–6.2)\) or 0.94–1.07. The other values in this table are found similarly. The superiority of the Hermite model seems clear.

Finally, note that for “hardening” cases (with narrower-than-Gaussian tails), there is no reason to question maximum entropy models. In these cases, they should yield similar results to Hermite models, which take the equivalent Gaussian fractile, \(u(x)\) in Eqn. 28, as a cubic polynomial in \(x\). Indeed, perhaps the main virtue of the Hermite model is its dual nature. In the general case where \(N\) moments are known, it models \(u(x)\) as an \((N-1)\)-order polynomial in the hardening case, and \(x(u)\) as an \((N-1)\)-order polynomial for softening, broader-than-Gaussian cases.

REFERENCES
L-Moments for Transformed Gaussian Variables

Consider now a new variable \( X = g(U) \), a functional transformation of a standard normal variable \( U \). Further assume that \( g \) is a monotonically increasing function; i.e., \( dg/dU \) is always positive. In this case, there is a one-to-one mapping between the CDF (or PDF) of \( X \) and \( U \):

\[
F(x) = \Phi(u) \quad f(x)dx = \Phi(u)du
\]

in which \( x = g^{-1}(u) \); i.e., the unique \( u \) value for given \( x \). Substituting this result into Eqn. 5,

\[
\lambda_n = \int_{-\infty}^{\infty} x \cdot w_n[F(x)]f(x)dx = \int_{-\infty}^{\infty} g(u) \cdot w_n[\Phi(u)]\Phi(u)du
\]

\[
= E\{g(U)w_n[\Phi(U)]\}
\]

Notably, this expected value is of the same form as Eqn. 11, now with \( U \) replaced by \( g(U) \). This simplifies calculations, particularly when the \( g \) function is a sum of terms: due to the linearity of the expectation operator, each of these terms can be handled separately. In particular, for \( X = u + c|U|^{m-1}U \) as in Eqn. 27, this result yields

\[
\tau_4 = \frac{\lambda_4[x]}{\lambda_2[x]} = \frac{\lambda_4[U] + c\lambda_4[|U|^{m-1}U]}{\lambda_2[U] + c\lambda_2[|U|^{m-1}U]}
\]

This is the basis for the rightmost column of results in Table 2. All numerical values have been computed with the public-domain solver Wolfram Alpha [11].

We can use a similar approach to calibrate Eqn. 23; i.e., choose \( b \) and \( c \) that yield a specified set of \( (\tau_3, \tau_4) \) values. Because \( \tau_3 \) and \( \tau_4 \) are unaffected by shifting and rescaling, it suffices to consider \( g(U) = U + b(U^2 - 1) + cU^3 \). If \( g(U) \) remains monotonic,\(^1\) L-moments can be computed by separately considering each term:

\[
\lambda_n = E[Uw_n(U)] + bE[(U^2 - 1)w_n(U)] + cE[U^3w_n(U)]
\]

in terms of the weight functions \( w_n(u) \) in Eqns. 32–34. Since \( w_1=1 \), \( \lambda_1=m_k \). For \( n=2 \) and \( 4 \), the weight functions \( w_n \) are odd so that the \( b \)-dependent term in Eqn. 38, \( E[(U^2 - 1)w_n(U)] \), vanishes. Thus, the results for \( \lambda_2, \lambda_4 \), and hence \( \tau_4 \), are independent of \( b \):

\[
\tau_4 = \frac{\lambda_4[U] + c\lambda_4[U^3]}{\lambda_2[U] + c\lambda_2[U^3]} = \frac{1 + Ac}{1 + Bc}
\]

\(^1\)To determine whether a cubic transformation \( g(U) \)—e.g., Eqn. 14 or Eqn. 23—remains monotonic, it is convenient to consider \( dg/dU=0 \) and require that the resulting quadratic equation have no real roots. This leads to the requirement that \( b^2 < 3c \) in Eqn. 23, and \( c^2 < 3c(1-3c) \) in Eqn. 14.

APPENDIX 1: SUPPORTING THEORY

Equation 11 shows the general integral expression for the L-moments, \( \lambda_n \), of a standard normal variable \( U \). To simplify this result, the cumulative distribution function \( \Phi(u) \) can be expressed as \( 0.5[1+\text{erf}(u/\sqrt{2})] \), in terms of the error function \( \text{erf}(x) \). Substituting this result into Eqns. 6–9, the weights in the Gaussian case become

\[
w_1(u) = 1; \quad w_2(u) = \text{erf} \left( \frac{u}{\sqrt{2}} \right) \quad (32)
\]

\[
w_3(u) = 1.5 \text{erf} \left( \frac{u}{\sqrt{2}} \right) - 0.5 \quad (33)
\]

\[
w_4(u) = 2.5 \text{erf} \left( \frac{u}{\sqrt{2}} \right) - 1.5 \text{erf} \left( \frac{u}{\sqrt{2}} \right) \quad (34)
\]

These weights are then used to evaluate the L-moments of \( U \) in Eqns. 12–13.

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Kurtosis: $\alpha_4 = \frac{E[X^4]}{E[X^2]^2}$

<table>
<thead>
<tr>
<th>Model of $X$:</th>
<th>Kurtosis:</th>
<th>L-Kurtosis:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U + cU \mid U$</td>
<td>$3 + 32\beta c + 90c^2 + 192\beta c^3 + 105c^4$</td>
<td>$0.06917047 + 0.316418c$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.5641896 + 0.818310c$</td>
</tr>
<tr>
<td>$U + cU^3$</td>
<td>$3 + 60c + 630c^2 + 3780c^3 + 10395c^4$</td>
<td>$0.06917047 + 0.807862c$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.5641896 + 1.410474c$</td>
</tr>
<tr>
<td>$U + cU^3</td>
<td>U$</td>
<td>$3 + 192\beta c + 5670c^2 + 184320\beta c^3 + 2027025c^4$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.5641896 + 2.77324c$</td>
</tr>
<tr>
<td>$U + cU^5$</td>
<td>$3 + 420c + 62370c^2 + 8108100c^3 + 654729075c^4$</td>
<td>$0.06917047 + 4.81779c$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.5641896 + 6.06504c$</td>
</tr>
</tbody>
</table>

Table 2. Moments and L-Moments for various transformations of a standard normal variable $U$. Note that $\beta = \sqrt{2}/\pi$ in these results.

in which

$$\tau_{4,\text{g.s.}} = \frac{\lambda_4[U]}{\lambda_2[U]} = 0.1226; \quad A = \frac{\lambda_4[U^3]}{\lambda_4[U]} = 11.68; \quad B = \frac{\lambda_2[U^5]}{\lambda_2[U]} = 2.5$$

Applying Eqn. 38 with $n=2$ and $3$, the L-skewness $\tau_3$ is

$$\tau_3 = \frac{\lambda_3}{\lambda_2} = \frac{bE[(U^2 - 1)w_3(U)]}{\lambda_2[U] + c\lambda_2[U^3]} = \frac{0.997b}{1 + Bc}$$

Notably, Eqns. 39–41 yield explicit results for $b$ and $c$, as given in Eqn. 25.

**Exact Moments for Hermite Transformation**

We consider here the exact skewness and kurtosis of the Hermite transformation model in Eqn. 14. Considering the rescaled case in which $X = U + c_1H_2(U) + c_4H_3(U)$, we have $m_X = 0$ and the higher central moments

$$E[X^2] = 1 + 2c_2^2 + 6c_4^2$$

$$E[X^3] = 6c_3 + 36c_3c_4 + 8c_3^3 + 108c_3c_4^2$$

$$E[X^4] = 3 + 24c_4 + 60c_3^2 + 252c_4^2 + 576c_3c_4 + 1296c_4^3 + 60c_3^4 + 2232c_4^2 + 3348c_4^4$$

Finally, the skewness and kurtosis are found from these results as $\alpha_3 = E[X^3]/E[X^2]^1.5$ and $\alpha_4 = E[X^4]/E[X^2]^2$. In the symmetric case $c_3 = 0$, and Eqn. 16 is found.