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Random Models of Second-Order Waves
and Local Wave Statistics

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Abstract

We consider second-order random models of ocean waves at arbitrary water
depths. We derive convenient new analytical results for wave moments, and show
results for crests and other local wave statistics. Theoretical predictions are com-
pared with observed wave tank results in extreme seas.

Introduction

Nonlinear hydrodynamic effects are of growing interest for ocean structures and vessels.
This has spurred development of efficient methods to estimate statistics of second-order
hydrodynamic models (e.g., Winterstein et al., 1994). Here we apply these to one of
the most fundamental nonlinearities in ocean engineering: the wave elevation \( \eta(t) \) at a
fixed spatial location.

Linear wave theory results in a Gaussian model of \( \eta(t) \). This ignores the marked
asymmetry of \( \eta(t) \): wave crests that systematically exceed subsequent troughs. This
has several practical implications: (1) asymmetric waves are more likely to strike decks
of offshore platforms, particularly older Gulf-of-Mexico structures with fairly low decks;
and (2) unusually large dynamic response has been found in high, steep waves that may
not follow linear theory.

Second-order random wave models are not new; indeed, they have been a research
topic for more than 30 years (e.g., Longuet-Higgins, 1963) and remain so today (e.g.,
Marthinsen and Winterstein, 1992; Hu and Zhao, 1993; Vinje and Haver, 1994). However,
they have not yet entered common offshore engineering practice, which applies either (1) random linear (Gaussian) waves or (2) regular Stokes waves that fail to
preserve \( S_\eta(\omega) \), the wave power spectrum. Several drawbacks to second-order random
waves may be suggested: (1) they omit potentially important higher-order effects; and
(2) convenient statistical analysis methods for second-order models are often lacking.
We seek to address both concerns here—the first through systematic comparison of
theory with observed wave tank results in extreme seas. The second issue is met by
fitting new analytical results for wave moments, and using these to construct simple
Hermite models of extreme crests.

Statistics of Second-Order Models

Our “input” is the first-order, Gaussian wave process \( \eta_1(t) \) from linear theory. The
standard Fourier sum for \( \eta_1(t) \) is then \( \text{Re} \sum C_k \exp(i \omega_k t) \), in which \( C_k = A_k \exp(i \phi_k) \) in
terms of Rayleigh distributed amplitudes, $A_k$, and uniformly distributed phases $\phi_k$. The resulting “output” is $x(t)=x_1(t)+x_2(t)$, in which

$$x_1 = \text{Re} \sum_k C_k H_k e^{i\omega_k t} ; \quad x_2 = \text{Re} \sum_k \sum_l C_k C_l [H_{kl}^+ e^{i(\omega_k+\omega_l)t} + H_{kl}^- e^{i(\omega_k-\omega_l)t}]$$  (1)

Here the transfer function $H_k$ describes first-order effects, while $H_{kl}^+$ and $H_{kl}^-$ reflect second-order effects at sums and differences of all wave frequencies ($\omega_k \pm \omega_l$). In our case $x(t)$ is the second-order wave itself, for which $H_{kl}^+$ and $H_{kl}^-$ are given analytically (e.g., Marthinsen and Winterstein, 1992), and $H_k=1$. The same analysis applies more generally to the diffracted wave, applied force and response of large-volume structures, with numerical $H_k$ and $H_{kl}$ estimates from second-order diffraction (Winterstein et al., 1994).

Because $x(t)$ is non-Gaussian, interest focuses on its skewness $\alpha_3$ and kurtosis $\alpha_4$. In terms of the significant wave height $H_S=\eta_1$ and peak spectral period $T_P$, these are

$$\alpha_3 \sigma_x^3 = m_{31}(T_P) H_S^4 + m_{33}(T_P) H_S^6$$  (2)

$$\alpha_4 - 3 \sigma_x^4 = m_{42}(T_P) H_S^6 + m_{44}(T_P) H_S^8$$  (3)

The $m_{ij}(T_P)$ are “response moment influence coefficients,” the contribution to response moment (cumulant) $i$ due to terms of order $O(x_j^i)$. In general these are conveniently calculated from Kac-Siegert analysis (Eqs. 12–15, Winterstein et al., 1994). We assume here the spectrum of $\eta(t)$ is of the form $H_S^2 T_P f(\omega T_P)$, so that $\eta(t)$ scales in amplitude with $H_S$ and in time with $T_P$.

It is useful to define the unitless wave steepness $s_p=H_S/L_P$, in which the characteristic wave length $L_P=g T_P^2/2\pi$ uses the linear dispersion relation. For deep-water waves the coefficients $m_{ij}(T_P)$ are proportional to $L_P^j$. Retaining leading terms in $s_p$ from Eqs. 2–3:

$$\alpha_3 = k_3 s_p \quad \alpha_4 - 3 = k_4 \alpha_3^2$$  (4)

In particular, for a JONSWAP wave spectrum with peakedness factor $\gamma$, we have fit the following $k_3$ and $k_4$ expressions to results for a wide range of depths:

$$k_3 = \frac{\alpha_3}{s_p} = 5.45\gamma^{-0.84} + 1.35 \left( \frac{d}{L_P} \right)^{-1.22} \quad k_4 = \frac{\alpha_4 - 3}{\alpha_3^2} = 1.41\gamma^{-0.00}$$  (5)

The second term in this result for $\alpha_3$ reflects the effect of a finite water depth $d$: in shallower water $\alpha_3$ grows, as the waves begin to “feel” the bottom.

Note also that while the skewness varies linearly with steepness, the kurtosis varies quadratically. This suggests that nonlinear effects will be most strongly displayed by the skewness, and hence by the wave crests rather than the total peak-to-trough wave height. This second-order model may less accurately predict kurtosis, however, as higher-order omitted effects may be of the same order of magnitude.

**Numerical Results**

Figure 1 compares skewness predictions with results from wave tank tests. The tests include 18 large seastates (target $H_S=14.5m-15.5m$), each over 3 hours in length, at water depths exceeding 300m. Figure 1 shows the resulting skewness and steepness in each hour of each test. While there is considerable scatter, regression on these data
gives the estimated slope $k_3=5.50$, remarkably close to Eq. 5 with $\gamma=1$. The scatter in Figure 1 is also consistent: the observed $\sigma_{\theta_3}$ is found well-predicted by simulated hourly segments of second-order seastates. Figure 2 shows kurtosis estimates to deviate, however. The data yield the estimate $k_4=4.2$, roughly 3 times the value in Eq. 5 regardless of $\gamma$. This again supports theory, which suggests that unlike the skewness, the kurtosis may be notably affected by unmodelled, higher-order effects.

Wave Crests. Figure 3 shows the observed distribution of crest heights. These results combine six seastates with the same target spectrum, and hence give roughly 20 hours of similar wave conditions. As expected the Rayleigh model, based on linear theory, is significantly unconservative. An alternative empirical model (Haring et al, 1976) offers an improvement, but only mildly changes the Rayleigh for these deep-water conditions. Better agreement is found from a Non-Gaussian (Hermite) model, which uses a cubic distortion of the normal process (and hence its Rayleigh peaks) to match $\alpha_3$ and $\alpha_4$ (Winterstein et al, 1994). Here estimates of $\alpha_3$ and $\alpha_4$ use Eq. 5, tripling its $k_4$ value to reflect unmodelled effects. These give excellent agreement with observed moments, though still somewhat unconservative crest predictions at higher levels.

Local Wave Characteristics. Figure 4 shows the conditional mean and standard deviation of crest height, given the peak-to-trough wave height. Again the data use 20 hours of wave tank studies, all with the same target wave spectrum. Also shown are corresponding estimates from simulation of Gaussian and Non-Gaussian (second-order) models. Due to the symmetry of the Gaussian model, its crests are on average 50% of the total wave height. The data shows systematically larger crests. The second-order model is found to predict this vertical asymmetry quite accurately.

We may also consider horizontal wave asymmetry; do crest front periods—during which $\eta$ increases from its mean level to a crest—differ statistically from subsequent crest back periods? This temporal asymmetry is not predicted by either the Gaussian or second-order model. It is difficult, however, to find this asymmetry in the data: Figure 5 shows observed crest fronts to be quite close to 50% of the total crest period. Finally, Figure 6 shows the variation of wave period with crest height. All results show the same trend of increasing periods over small-to-moderate heights, and roughly constant period at large heights. The Non-Gaussian model appears to somewhat better predict results at larger crest height levels.

References


**Figure 1:** Skewness: Theory vs Data

**Figure 2:** Kurtosis: Theory vs Data

**Figure 3:** Crest Heights: Theory vs Data

**Figure 4:** Variation of Crest Height with Wave Height

**Figure 5:** Variation of Crest Front Period with Total Crest Period

**Figure 6:** Variation of Crest Height with Wave Period