MODELLING SHALLOW-WATER WAVES:
TRUNCATED HERMITE MODELS AND THE SHALLOWWAVE ROUTINE

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ABSTRACT
Statistical modelling of ocean waves is complicated by their nonlinearity, which leads in turn to non-Gaussian statistical behavior. While non-Gaussianity is present even in deep-water applications, its effects are especially pronounced as water depths decrease. We apply two types of wave models here: (1) local models of extreme wave heights/periods and breaking limits, and (2) random process models of the entire non-Gaussian wave surface. For the random process approach, we derive a new “truncated” Hermite model, which can reflect four moments and both upper- and lower-bound limiting values due to breaking and finite-depth effects. Results are calibrated and compared with an extensive model test series, comprising up to 23 hrs of histories across 19 seastates, at depths from 15–67m (full scale).

INTRODUCTION
Statistical modelling of ocean waves is complicated by their nonlinearity, which leads in turn to non-Gaussian statistical behavior. While non-Gaussianity is present even in deep-water applications, its effects are especially pronounced as water depths decrease.

Our interest here lies particularly with offshore wind turbines, commonly placed in relatively shallow water depths. Extreme wave loads on these turbines will likely be affected by non-Gaussian behavior, and possibly breaking conditions as well. This has motivated an extensive series of shallow-water model tests, conducted by Marintek and sponsored by Statoil [1].

Two types of wave models are considered here:

1. Local models of extreme wave heights, \( H \), and periods, \( T \), based on the most recent breaking models of Goda [2].
2. Non-Gaussian models of the wave surface process \( \eta(t) \), using truncated versions of Hermite models (e.g., [3, 4]).

Approach 1 is motivated by the ongoing use of such breaking criteria in coastal engineering, and offshore wind turbines in particular (e.g., [5]). Note in that reference that associated loads are estimated through the direct use of loads data, from model tests on a monopile foundation.

Approach 2, the random process model, is suggested by our relative success in using Hermite models in deep-water applications (e.g., [6]). If successful, the simulation of the wave surface \( \eta(t) \) is hoped to be combined with an appropriate kinematics model, to directly predict loads without model test results.

Approach 2 requires several technical developments. First, a new truncated version of the Hermite model is developed, to preserve both the first four statistical moments of \( \eta(t) \) and the upper- and lower-bound limits, \( \eta_{\text{max}} \) and \( \eta_{\text{min}} \), to reflect breaking and finite-depth effects. This has been implemented in the new simulation routine ShallowWave [7].

Beyond this, our standard deep-water simulation relies on second-order random waves. While second-order theory has the ability to include a finite water depth, we find here that it is unable to accurately predict the strong non-Gaussianity that arises...
in shallow water. Therefore, we develop new wave statistics—
e.g., corrections on skewness predicted by second-order theory—
as a function of water depth.

**DESCRIPTION OF MODEL TESTS**

A series of shallow-water wave tests were performed at the
“lilletanken” at Marintek. The model length scale has been cho-
osen as 1:81, implying a time scale of 1:9. Each test is roughly
2.3hr full scale. There were up to 10 realizations performed of
each test; i.e., up to 23 hrs of data are available for the various
seastate conditions.

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Table 1. Model test numbers and characteristics.

Table 1 shows the range of tested seastates. For each of 5
$T_p$ values, tests were performed at various scalings of $H_S$, aiding
the study of the onset of breaking. Note too that the highest $H_S$
values tested, 8.5m and 11.3m, are roughly representative of ALS
and ULS (100- and 10,000-year) conditions at the North Sea site
of Statoil’s Hywind offshore turbine.

The tests considered here had a sloping bottom of 1:20, or
2.8 degrees. Figure 1 shows the placement of the wave probes.
These ranged from the deepest probe (#2) at full-scale depth
d=67m, to the shallowest probe (#7) at d=15m.

**RANDOM MODELS OF EXTREME WAVES AND
BREAKING**

In a recent study [2], Goda has summarized and updated his
extensive work on wave breaking statistics. As a function of the
water depth $d$ and infinite-depth wave length $L_0=gT^2/(2\pi)$, Eq.
6 of [2] gives the “wave breaker index” formula

$$\frac{H}{L_0} = A \left\{ 1 - \exp\left[ -\beta \left( \frac{d}{L_0} \right) \right] \right\} : \beta = 1.5\pi (1 + 11 \tan^{4/3} \theta) \quad (1)$$

In Eqn. 1, Goda suggests the value $A=0.17$ as the deep-water
limiting steepness. The coefficient $\beta$ includes the effect of the
beach slope angle $\theta$. Equation 1 is offered in [2] as a correction
to Goda’s earlier work, which is said to overestimate the slope
effect term (the coefficient “11” in Eqn. 1 replaces Goda’s earlier
value of “15”). For our tests, $\tan\theta=0.05$ so that Eqn. 1 yields
$\beta=5.667$.

Figure 2 shows $(H, T)$ pairs observed at various probes for
seastates 2115–2116, with $T_p=13.5s$ and the largest 2 $H_S$ values
($H_S=8.5$ and 11.3m). Numerical $(H, T)$ values are reported in
model scale. Little difference is found between these two $H_S$
levels, because breaking conditions are being approached. Three
depths are considered: probe 2 (deepest at $d=67m$), probe 10
(intermediate depth $d=26m$), and probe 7 (shallowest at $d=15m$).
Results for other depths and seastates are similar [7].

Results are compared to Eqn. 1 both with $A=17$, the clas-
sical value suggested by Goda, and $A=20$. The classical result
appears fairly accurate; however, one may argue for a slightly
larger value of $A$ than 0.17 at the shallower depths. Note, too,
that our slope of 1:20 is rather steep, and $A=17$ may yield greater
agreement for shallower slopes.
RANDOM PROCESS MODELS 1: SKEWNESS

The main effect of nonlinearity is to produce asymmetric waves, whose crests systematically exceed their troughs. The most important statistical measure of wave nonlinearity is therefore the skewness value, $\alpha_3$. In infinite water depths, $\alpha_3$ has been found fairly accurately predicted by second-order theory (e.g., [8, 9]). Assuming a JONSWAP spectrum with peakedness factor $\gamma$, the second-order skewness prediction is roughly [7]

$$\alpha_{3, second-order} = k_3 s_p; \quad s_p = \frac{H_s}{L_p} \quad k_3 = \frac{5.47}{l^{0.087}}$$

in which $L_p$ is the finite-depth wave length associated with $T_p$. This fitted result is similar to that in [6], slightly modified to return the exact result when $\gamma=1$ [10].

Second-order theory can also be used to estimate $\alpha_3$ in shallow water. Unfortunately, its depth-dependence is found inadequate. This is shown in Fig. 3, which compares observed $\alpha_3$ values with depth-dependent predictions from second-order theory (e.g., in tests 2109 and 2112). While theory predicts the deep-water asymptote fairly well, it fails to follow the sharp increase in $\alpha_3$ as depths become shallow.

This motivates our study of $\alpha_3/\alpha_{3, second-order}$, that is, we normalize observed $\alpha_3$ test values by the infinite-depth prediction in Eqn. 2. Figure 4 shows that the resulting data follow a strong systematic trend. (Note that we exclude here seastates with steepness $s_p < 0.3$, which are relatively uninteresting for our purposes.) The net result is

$$\alpha_3 = \frac{5.47 s_p}{l^{0.087}} \left[1 + \left(\frac{x}{0.11}\right)^{-3.5}\right]; \quad s_p = \frac{H_s}{L_p}; \quad x = \frac{d}{L_p}$$

This fit appears particularly accurate in the shallow-water range, where $\alpha_3$ levels are largest (and hence this correction most important). The greatest error in Eqn. 3 appears at larger depths, where our data suggests a slightly lower $\alpha_3$ value than given by second-order theory. Nonetheless, we retain the second-order theory result here, due to its widespread acceptance in deep-water applications.

RANDOM PROCESS MODELS 2: KURTOSIS

Figure 5 shows the seastate kurtosis, $\alpha_4$, for various tests and depths, plotted not versus depth but rather versus its corresponding $\alpha_3$ value. This is motivated by noting that various non-Gaussian models are of the form

$$\alpha_4 = 3 + k\alpha_3^n$$

In particular, for a number of common probability models (including the leading behavior of second-order waves), Eqn. 4
Figure 3. Estimated skewness, $\alpha_3$, from various tests at various depths.

Figure 4. Skewness correction, $\alpha_3/\alpha_{3,2nd-order}$, where $\alpha_{3,2nd-order}$ is second-order prediction at infinite depth (Egn. 2).

Figure 5. Estimated kurtosis, $\alpha_4$, from various tests at various depths.

seastates are unusually large—if they are found to be of interest, Eqn. 5 will likely overestimate $\alpha_4$.

**RANDOM PROCESS MODELS 3: FULL DISTRIBUTION**

We now assume knowledge of $\alpha_3$ and $\alpha_4$, together with the upper- and lower-bound limits, $\eta_{\text{max}}$ and $\eta_{\text{min}}$, on the wave surface $\eta(t)$ due to breaking and finite-depth effects. To create a truncated Hermite model of $\eta(t)$, the following steps are taken:

1. We first assume the standard Hermite model:

   $x(t) = \frac{\eta(t) - \eta_0}{\sigma_{\eta}} = x_0 + \kappa\{u + c_3(u^2 - 1) + c_4(u^3 - 3u)\}$  \(6\)

2. We then calculate the truncated values, $x_{\text{max}}$ and $x_{\text{min}}$, of the standardized response:

   $x_{\text{max}} = \frac{\eta_{\text{max}} - \eta_0}{\sigma_{\eta}}$, $x_{\text{min}} = \frac{\eta_{\text{min}} - \eta_0}{\sigma_{\eta}}$

3. We assume values of $x_0$ and $\kappa$ in Eqn. 6.

4. To reflect $x_{\text{max}}$ and $x_{\text{min}}$, we truncate Eqn. 6:

   $x_t = g(x) = \begin{cases} 
   x_{\text{max}} \left\{1 - \exp\left[-\left(\frac{x}{x_{\text{max}}}\right)^p\right]\right\}^{1/p} & x \geq 0 \\
   x_{\text{min}} \left\{1 - \exp\left[-\left(\frac{x}{x_{\text{min}}}\right)^p\right]\right\}^{1/p} & x < 0
   \end{cases}$  \(7\)

This result will return $x_t \approx x$ for small $x$, while the truncation limits are approached as $x \to \pm\infty$. The parameter $p$ reflects how quickly the truncation is approached; higher $p$ values lead to sharper, more “corner-like” behavior.
5. A standard Newton-Raphson routine is used to find \( c_3 \) and \( c_4 \) in Eqn. 6 so that the skewness and kurtosis in Eqn. 7 have the desired values.

6. Finally, we introduce the shift and rescaling factors, \( x_0 \) and \( \kappa \), to preserve the desired mean and variance of \( \eta(t) \).

Note that the standard, untruncated Hermite model would require only steps 1, 3, 5, and 6. Also, in the standard Hermite case these steps would be required only once. For the truncated case, however, the shifting/rescaling in step 6 will generally alter the upper- and lower-bound values \( \eta_{\text{max}} \) and \( \eta_{\text{min}} \). Thus, following step 6 one typically returns to step 3, and iterates until \( x_0 \) and \( \kappa \) (and hence \( \eta_{\text{max}} \) and \( \eta_{\text{min}} \)) converge. This implies a nested set of iterations: an inner Newton-Raphson iteration for \((c_3, c_4)\), and an outer iteration to find \( x_0 \) and \( \kappa \). In our practical experiences this has not been found numerically burdensome; the outer iteration typically requires only 5-10 steps, and proceeds monotonically toward convergence in the scalar quantity \( E[\eta^2] \) (to unity).

Finally, note also that the end result, while complicated, remains a functional transformation of a Gaussian process:

\[
u \rightarrow x \text{ (Eqn. 6)} \rightarrow \eta = m_\eta + \sigma_\eta x_i \tag{8}\]

Symbolically, \( \eta(t) = g(u(t)) \), a memoryless transformation of a Gaussian process. This permits various statistics of \( \eta \) to be estimated analytically; e.g., the probability distribution \( P[\eta(t) \leq y] \) is equal to \( \Phi(g^{-1}(y)) \), in which \( \Phi \) is the standard normal CDF. Of course, one need not invert \( g \) to plot \( p = P[\eta(t) \leq y] \); one merely plots \( p = \Phi(u) \) versus \( y = g(u) \).

Figure 6 compares this predicted distribution for test 2103 with the 3 realizations available for this test. These model-scale predictions use \((\alpha_3, \alpha_4)\) from the tests and \((\eta_{\text{max}}, \eta_{\text{min}}) = (0.090 \text{m}, -0.065 \text{m})\). Results are rather accurate over a range of depths.

If the Hermite transformation model, \( \eta(t) = g(u(t)) \), holds at all times \( t \), it should apply at the times when wave crests occur. Thus, a result analogous to Eqn. 8 can be used to estimate \( C_p \), the \( p \)-fractile of a random crest of \( \eta(t) \):

\[
C_p = g(R_p) = m_\eta + \sigma_\eta x_i(R_p) ; \tag{9}
\]

\[
R_p = \sqrt{-2 \ln(1-p)} \tag{10}
\]

In Eqn. 10, \( R_p \) is the \( p \)-th fractile of a Rayleigh variable, which models a random peak of a narrow-band, standard normal process.

Figure 7 compares observed crest distributions with predictions from Eqns. 9–10. Compared to its predictions of the wave elevation distribution (Fig. 6), the same truncated Hermite model tends to somewhat underestimate the wave crest distributions. It is difficult to argue that a different \( g \) function should be used, in view of the accuracy of the marginal distribution comparisons.
in Fig. 6. Errors in crest modelling would suggest the fault lies with the functional transformation model; e.g., velocities of \( \eta \) may be more strongly dependent on \( \eta \) values than the transformation model predicts. This may be investigated further, and/or fixed by some empirical correction.

**EXAMPLE: SIMULATING 100-YEAR CONDITIONS**

Finally, we use the foregoing results to simulate 100-year seastate conditions at water depth \( d=25m \). The seastate conditions are assumed to be

**100-Year Seastate:** \( H_S=9m, \ T_P=14s, \ L_P=200m \)

The \( L_P \) values follow from the depth-dependent dispersion relation. As with the model tests, our simulations use a JONSWAP spectrum with \( \gamma=3.3 \).

From our test series, probe 11 is the most relevant, located precisely at \( d=25m \) in full scale. Test 2115 is the closest to the 100-year conditions above; at probe 11 we find the statistics

**Test 2115:** \( H_S=8.51m, \ T_P=13.5s, \ \sigma_\eta=2.19m, \ \alpha_3=0.34, \ \alpha_4=3.23, \ \eta_{max}=10.53m, \ \eta_{min}=-6.82m \)

Note that we also have empirical fits to \( \alpha_3 \) and \( \alpha_4 \). From Eqn. 3 with \( \gamma=3.3 \) and Eqn. 5:

\[
\alpha_3 = 0.36 ; \quad \alpha_4 = 3.25
\]  

We are encouraged to find that the values in Eqn. 11 are close to those observed in Test 2115. We therefore adopt them here for our simulations. We consider how to estimate \( \eta_{max} \) directly from theory in Appendix 1.

Figure 8 shows \((H, T)\) pairs in upcrossing waves, concatenating results across the 10 realizations of Test 2115 at probe 11. Results here are shown in full scale, rather than model scale. As in Fig. 2, for reference we also show predictions from Goda’s result in Eqn. 1, with both \( A=.17 \) and \( A=.20 \). As before, Goda’s result is found relatively accurate.

In contrast, Fig. 9 shows \((H, T)\) pairs in upcrossing waves, based on 30 hours of simulation of 100-year conditions (\( H_S=9m, \ T_P=14s, \ \alpha_3=.36, \ \alpha_4=3.25 \)). The Goda predictions are again as in Fig. 8. In comparison with these, there are clearly far fewer high-steepness events than shown by the model tests in Fig. 8.

Recall that the foregoing simulations have assumed a JONSWAP spectrum with peakedness factor \( \gamma=3.3 \). We consider here the accuracy of this assumption. We may imagine that in these shallow water tests, some energy is transferred away from long, low-frequency waves, resulting in a spectrum that is relatively broader in its high frequency content. Figure 10 confirms that this is the case. It shows the empirical wave spectrum estimated from one realization of test 2115 at wave probe 11. Compared with the JONSWAP used in the simulations, it is somewhat less
peaked and more broad in the upper frequency range. We may then ask whether this enhanced high frequency content can serve to “fill in” the high-steepness region of the \((H, T)\) scattergram in Fig. 9.

The simulation program ShallowWave [7] permits the user to enter a numerically-defined spectrum. Figure 11 shows the result of using this numerically-based PSD option in the simulation. Specifically, it takes as input the empirical spectrum of Test 2115 (realization 1), as shown in Fig. 10. By including its somewhat broader-than-JONSWAP upper tail, the result is to achieve many more \((H, T)\) pairs near the high-steepness limits. This gives far greater agreement with the simulation results in Fig. 8. This shows the need for a spectral model that reflects shallow-depth effects. One such model, the TMA spectrum, is discussed below.

**THE TMA SPECTRUM**

The TMA spectrum modifies a reference spectrum \(S_{REF}(\omega)\), assumed to apply at infinite depth, by a depth-dependent correction factor \(\phi(\omega, d)\):

\[
S_{TMA}(\omega, d) = S_{REF}(\omega)\phi(\omega, d)
\]  

(12)

The original reference to \(\phi\) appears to be due to Kitaigordskii
et al [11]. From its Eqs. (3.3)-(3.4), \( \phi \) is defined implicitly:

\[
\phi(\omega, d) = \kappa^{-2}(\omega_d) \left[ 1 + \frac{2 \omega_d^2 \kappa(\omega_d)}{\sinh(2 \omega_d^2 \kappa(\omega_d))} \right]^{-1}
\]

(13)
in which

\[
\omega_d = \frac{\omega}{\omega_0}; \quad \omega_0 = \sqrt{\frac{g}{d}}
\]

(14)
is a unitless frequency measure, and \( \kappa(\omega_d) \) is given implicitly by

\[
\kappa \tanh(\omega_d^2 \kappa) = 1
\]

(15)

This result is also shown by Massel [12] (Eqs. 3.65-3.68).

The asymptotic behavior of \( \phi \) for small frequencies is

\[
\phi(\omega) \sim \frac{1}{2} \omega_d^2 = \frac{1}{2} \left( \frac{\omega}{\omega_0} \right)^2; \quad \omega \to 0
\]

(16)

Clearly, \( \phi \) serves to reduce the energy content of long (low-frequency) waves. Up to the mid-frequency range for which Eqn. 16 holds, \( \phi \) is proportional to \( \omega^2 \) so that the spectrum decays only like \( \omega^{-3} \), rather than the original \( \omega^{-5} \) decay of the original JONSWAP form. Of course, in the high-frequency limit \( \phi \to 1 \), and the \( \omega^{-5} \) decay rate is recovered.

**Standard Deviations Predicted By the TMA Spectrum**

A common way to specify seastate conditions—e.g., for design or model tests—is through \( H_{\text{sw}} \), the significant wave height that would be found if the water depth were infinite. (The area under the deep-water spectrum \( S_{\text{REF}}(\omega) \) in Eqn. 12 would be \( (H_{\text{sw}}/4)^2 \).) We study here how the TMA spectrum alters the standard deviation, \( \sigma_\eta \), of the wave process as a function of water depth.

For the TMA spectrum, the standard deviation, \( \sigma_\eta \), satisfies

\[
\sigma_\eta^2 = \int_{all \omega} S_{\text{TMA}}(\omega) d\omega = \int_{all \omega} S_f(\omega) \phi(\omega) d\omega
\]

(17)

We assume here that \( S_{\text{REF}} \) is a JONSWAP spectrum (hence \( S_f \)), with constant peakedness factor \( \gamma \). The spectral shape then scales directly with \( \omega_p \), the peak frequency \( \omega_p = 2\pi/T_p \):

\[
S_f(\omega) = \frac{H_{\text{sw}}^2}{\omega_p} \frac{\omega}{\omega_p} f \left( \frac{\omega}{\omega_p} \right)
\]

(18)

Substituting Eqn. 18 into Eqn. 17, and noting that \( \phi \) is a function of \( \omega_d = \omega \sqrt{d/g} \):

\[
\sigma_\eta^2 = H_{\text{sw}}^2 \int_{all \omega} f \left( \frac{\omega}{\omega_p} \right) \phi(\omega) \sqrt{\frac{d}{g}} \frac{d\omega}{\omega_p}
\]

\[
= H_{\text{sw}}^2 \int_{all \omega} f(u) \phi(u \cdot \omega_p) \sqrt{\frac{d}{g}} du; \quad u = \frac{\omega}{\omega_p}
\]

This latter form shows that the normalized quantity, \( \sigma_\eta/H_{\text{sw}} \), depends only on a single dimensionless parameter, \( \omega_p \sqrt{d/g} \). The square of this quantity is \( \omega_d^2 d/g \), or simply \( k_w d \) in terms of the wavenumber, \( k_w \), in the infinite-depth dispersion relation \( \omega^2 = g k_w \). Equivalently, we may consider the governing parameter as \( d/L_{P\text{sw}} \), in which \( L_{P\text{sw}} \) is the infinite-depth wave length:

\[
\frac{\sigma_\eta}{H_{\text{sw}}} = g(x); \quad x = \frac{d}{L_{P\text{sw}}}; \quad L_{P\text{sw}} = \frac{g}{2\pi} T_p^2
\]

(19)

For each probe location, Fig. 12 plots the predicted value of \( \sigma_\eta \), from numerical integration of Eqn. 17, versus \( x = d/L_{P\text{sw}} \). The figure confirms that when plotted in this way, all TMA predictions fall along one curve. It also shows, for the \( \gamma=3.3 \) case considered, that this TMA curve is well-predicted by the simple result

\[
\sigma_\eta = \frac{H_{\text{sw}}}{4} \left[ 1 - 0.75 \exp(-7.5x) \right]; \quad x = \frac{d}{L_{P\text{sw}}}; \quad L_{P\text{sw}} = \frac{g}{2\pi} T_p^2
\]

(20)

Provided the TMA spectral model is accurate, this should be a convenient basis to specify \( \sigma_\eta \) in the moment-based Hermite model.

**TMA Spectrum Compared with Data**

It remains to study the accuracy of the TMA model. To illustrate the spectral evolution with depth, Fig. 13 shows spectral estimates at wave probes 2, 3, 4, 11, 7, and 6 (in order of decreasing depth). The largest differences arise in the mid-frequency range; e.g., from around \( f_p=0.07 \)Hz to around \( 2f_p=0.14 \)Hz. In this frequency range, the energy is systematically reduced with the water depth, as the TMA model would suggest.

Finally, we ask whether the TMA reduction factor, \( \phi \) in Eqn. 13, is supported by the data. To address this, it is convenient to note the multiplicative form of the TMA spectrum. Equation 12 implies that the ratio of wave spectra, observed at different depths \( d_1 \) and \( d_2 \) in the same wave conditions, depends not on \( S_{\text{REF}} \) but only on \( \phi \):

\[
\frac{S(\omega, d_1)}{S(\omega, d_2)} = \frac{\phi(\omega, d_1)}{\phi(\omega, d_2)}
\]

(21)

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Figure 12. Depth-dependent standard deviation, $\sigma_\eta(d)$, estimated from TMA spectrum across all probe locations for all model tests.

Figure 14 compares the observed spectral ratios, $S(\omega, d_1)/S(\omega, d_2)$, with those predicted by TMA. All results use the reference depth $d_2=67$m, i.e., the deepest depth for which observations are available (probe 2).

The results in Fig. 14 are a bit challenging to interpret. (Note in the figure that results at the same depth are shown in the same color; e.g., both predictions and measurements at $d_1=54$m are shown in red, at $d_1=40$m in green, and so forth.) It is useful to focus on the mid-frequency behavior; e.g., between $f_p=0.07$Hz and $2f_p=0.14$Hz. From Fig. 13, this appears the most critical range to model accurately. There is little evidence here that the TMA model follows the correct functional form; however, it may perhaps be used in its mean value (over this range) to predict the observed spectral ratio. Note too that the agreement with TMA may improve for a shallower slope than the rather steep (1:20) cases tested here.

CONCLUSIONS
1. With respect to local models of wave height $H$ and period $T$, Goda’s formula (Eqn. 1) reflects breaking conditions rather accurately. Perhaps a slightly larger $A$ value than 0.17, suggested by Goda, should be favored for shallower seastates (Fig. 2).
2. A truncated version of the Hermite model has been derived. This permits matching of wave skewness and kurtosis, $\alpha_3$ and $\alpha_4$, as well as of the upper- and lower-bound limits, $\eta_{\text{max}}$ and $\eta_{\text{min}}$, on the wave surface process $\eta(t)$. This has been implemented in the simulation routine ShallowWave. Results follow the empirical distribution of $\eta(t)$ rather well (e.g., Fig. 6).
3. Empirical results have been fit to estimate $\alpha_3$ and $\alpha_4$ for these shallow-water conditions (Eqn. 3 and Eqn. 5). These results appear to follow the data trends fairly accurately (Figs. 4–5).
4. To reproduce steep waves, it has been found important to have accurate wave spectral models (Fig. 9 vs Fig. 11). This includes the increasing removal of low-frequency energy as the water depth decreases (Fig. 13). The TMA spectral model attempts to model this. While the data do not support the TMA model in detail (Fig. 14), in the high-energy, mid-frequency range at least the average spectral reduction, $\phi$, from TMA is roughly consistent with the data.
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REFERENCES


APPENDIX I: PREDICTING MAXIMUM CREST ELEVATION

In the foregoing, we have provided analytical predictions of the depth-dependent standard deviation $\sigma_\eta(d)$ (Eqn. 20), the skewness $\alpha_3$ (Eqn. 3), and the kurtosis $\alpha_4$ (Eqn. 5). For the truncated Hermite model, the remaining parameters needed are the theoretical upper and lower bounds, $\eta_{\text{max}}$ and $\eta_{\text{min}}$. We consider here briefly how these quantities—$\eta_{\text{max}}$ in particular—can be estimated directly.

Depth-induced breaking criterion

As the water depth $d$ decreases, one may imagine that the maximum crest $\eta_{\text{max}}$ becomes proportionally less as well; i.e.,

$$\eta_{\text{max}} = \alpha d$$  \hspace{1cm} (22)

Indeed, Eqn. 22 with $\alpha=0.55$ has been suggested as a simple theoretical upper-bound limit on the maximum crest, $\eta_{\text{max}}$, to reflect breaking [13]. Figure 15 shows that a few observations at the shallowest depths exceed Eqn. 22 with $\alpha=0.55$. To envelope the data, a value of roughly $\alpha=0.75$ is shown to be needed. The value $\alpha=0.55$ is of course only approximate, and does not reflect the sloping depth. In particular, the slope here, 1:20, is relatively steep, so these tests may show enhanced shallow-water effects compared with the more gently sloping cases of most practical interest.
Figure 16. Maximum crest observed in test series versus observed wave standard deviation, $\sigma$, at each probe.

RMS-induced breaking criterion

Seeking other statistics that may “explain” the maximum crest, various parametric studies have been performed (e.g., versus steepness, Ursell number, etc.). Perhaps the simplest, and most successful, simply relates $\eta_{\text{max}}$ to the depth-dependent RMS value $\sigma_{\eta}(d)$. In particular, a simple upper bound of the form

$$\eta_{\text{max}} \leq 6\sigma_{\eta}(d)$$

(23)

is shown in Fig. 16 to envelope the data. Again, it should be noted that this result is specific to this dataset; i.e., a sloping bottom of 1:20.